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A CHARACTERIZATION OF LOCAL CONNECTEDNESS FOR GENERALIZED CONTINUA

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A topological space is called a generalized continuum if it is a connected, locally compact, separable, metric space. A topological space is called a semi-continuum if any two of its points lie in a sub-continuum of the space. It is well known that if X is a locally connected generalized continuum and U is an open connected subset of X, then U is a semi-continuum (in fact U is arcwise connected). In this paper (¹) we show that this property characterizes local connectedness for generalized continua. That is, if X is a generalized continuum with the property that each of its open connected subsets is a semicontinuum, then X is locally connected. A corollary of this result (²) answers a question raised in [2] (see also [1]). At the end of the paper we give an example which shows that this result does not hold for non-metric continua.

Definition 1. A topological space is said to be a *continuum* if it is a compact, connected, metric space.

Definition 2. A topological space X is said to be a *semicontinuum* if given any two points $x, y \in X$, there exists a continuum $K \subset X$ such that $x, y \in K$.

Definition 3. A topological space is said to be a *generalized continuum* if it is a connected, locally compact, separable, metric space.

Definition 4. A topological space X is said to be *locally connected at a point* $p \in X$ if given any neighborhood U of p, there exists a neighborhood V of p such that V is contained in the component of U containing p. If X is locally connected at each of its points, then X is said to be *locally connected*.

(¹) This research was conducted while the author was participating in an exchange of scientists administrated jointly by the Polish Academy of Sciences and the National Academy of Sciences (U. S. A.).

(²) In the original version of this paper the author proved only this corollary. The author is indebted to R. Duda for suggesting the more general theorem and also for greatly simplifying the example given at the end of the paper.

The following theorem is proved in [3] (theorem 12.1, p. 18). Convergence of sets is understood to be in the sense defined in [3].

THEOREM 1. *If a generalized continuum X fails to be locally connected at some point $p \in X$, then there exists a spherical neighborhood R with center p and an infinite sequence of distinct components $N_1, N_2, N_3, \ldots$ of $X \cap \mathrm{Cl}(R)$ converging to a limit continuum N which contains p and has no point in common with any of the continua $N_1, N_2, N_3, \ldots$*

Definition 5. A closed subset C of a generalized continuum X is said to be a *cutting* of X if $X \setminus C$ contains two open sets M_1 and M_2 such that $M_1 \cup M_2 = X \setminus C$ and $M_1 \cap M_2 = \emptyset$.

Definition 6. A cutting C of a generalized continuum X is said to *separate* two subsets X_1 and X_2 of X if $X_1 \cup X_2 \subset X \setminus C$ and the sets M_1 and M_2 of definition 5 may be chosen so that $X_1 \subset M_1$ and $X_2 \subset M_2$.

Definition 7. A disjoint family $\{C_a: a \in \Gamma\}$ of cuttings of a generalized continuum X is said to be *non-separated* if given any two cuttings C_a and C_β in the family and any two open subsets M_1 and M_2 of $X \setminus C_a$ such that $M_1 \cup M_2 = X \setminus C_a$ and $M_1 \cap M_2 = \emptyset$, it follows that either $C_\beta \subset M_1$ or $C_\beta \subset M_2$.

The following theorem is a corollary of theorem 2.2, p. 45 in [3]:

THEOREM 2. *If $\{C_a: a \in \Gamma\}$ is an uncountable, disjoint, non-separated family of cuttings of a generalized continuum X and $q \in X \setminus \bigcup \{C_a: a \in \Gamma\}$, then there exist cuttings C_a and C_β in the family such that C_a separates $\{q\}$ and C_β.*

THEOREM 3. *If X is a generalized continuum with the property that each of its open connected subsets is a semi-continuum, then X is locally connected.*

Proof. Suppose X with metric ϱ is as above and that X fails to be locally connected at some point $p \in X$. Then let the continua $N_1, N_2, N_3, \ldots$ and N be as in theorem 1 and let $q \neq p$ be another point of N. Let r be a positive real number such that $r < \varrho(p, q)$ and for every t, where $0 < t \leqslant r$, the set $U_t = \{x \in X: \varrho(x, p) < t\}$ has compact closure. For every real number t such that $0 < t \leqslant r$ we define the following sets:

$$
\begin{aligned}
U_t &= \{x \in X: \varrho(x, p) < t\},\\
P_t &= \text{the closure of the component of } U_t \text{ containing } p,\\
K_t &= P_t \setminus U_t,\\
Q_t &= \text{the component of } X \setminus P_t \text{ containing } q,\\
C_t &= \mathrm{Cl}(Q_t) \cap K_t.
\end{aligned}
$$

We now observe two rather obvious facts. First, by the choice of $N, N_1, N_2, N_3, \ldots$ and q, none of the sets C_t separates $\{p\}$ and $\{q\}$. Second, since for each t where $0 < t \leqslant r$ and for any $x \in C_t$ we have $\varrho(x, p) = t$, it follows that the C_t's are disjoint. Moreover, we claim that each of the sets C_t is a cutting of X. For suppose that this were not the case for some t with $0 < t \leqslant r$. Then the open set $X \setminus C_t$ would be connected and thus by hypothesis it would contain a continuum B such that $p, q \in B$. Since B contains q and meets P_t, it must contain a continuum which is irreducible with respect to these two properties, i.e. there is a subcontinuum A of B such that A is irreducible with respect to containing q and meeting P_t. Now for each positive number ε such that $\varepsilon < \varrho(q, P_t)$ let us define a set

$$V_\varepsilon = \{x \in X \colon \varrho(x, P_t) < \varepsilon\}$$

(note that since $P_t \subset P_r$ for every $0 < t \leqslant r$ and $q \in X \setminus P_r$, we must have $\varrho(q, P_t) > 0$). Also for each positive real number $\varepsilon < \varrho(q, P_t)$ we define a continuum

$E_\varepsilon =$ the closure of the component of $A \setminus \mathrm{Cl}(V_\varepsilon)$ containing q.

Then for each ε we have $E_\varepsilon \cap \mathrm{Cl}(V_\varepsilon) \neq \emptyset$ (if E is a component of an open subset U of a continuum A, then the closure of E meets the boundary of U). Thus for each ε we have $\varrho(E_\varepsilon, P_t) \leqslant \varepsilon$. Moreover, the E_ε's are nested and each of them contains q. Thus $\mathrm{Cl}\big(\bigcup \{E_\varepsilon \colon 0 < \varepsilon < \varrho(q, P_t)\}\big)$ is a subcontinuum of A which contains q and meets P_t. The irreducibility of A now allows us to conclude that $\mathrm{Cl}\big(\bigcup \{E_\varepsilon \colon 0 < \varepsilon < \varrho(q, P_t)\}\big) = A$. Since $E_\varepsilon \subset Q_t$ for every ε, the above implies that

$$A \subset \mathrm{Cl}(Q_t) \tag{1}$$

Now let $q_1 \in A \cap P_t$. Since $A \cap C_t = \emptyset$ and $A \subset \mathrm{Cl}(Q_t)$, we must have

$$A \cap K_t = \emptyset. \tag{2}$$

Thus $q_1 \in P_t \setminus K_t$, i.e. $q_1 \in U_t$. Let E be the component of $A \cap U_t$ containing q_1. Then $\mathrm{Cl}(E)$ meets the boundary (in A) of $U_t \cap A$, i.e. there exists $q_2 \in \mathrm{Cl}(E)$ which is on the boundary of $U_t \cap A$. But clearly $q_2 \in P_t$ and $q_2 \notin U_t$ ($\varrho(p, q_2) = t$), i.e. $q_2 \in A \cap K_t$. This contradicts (2) and establishes our claim. Thus each C_t is a cutting of X.

We now claim that the family $\{C_t \colon 0 < t \leqslant r\}$ is non-separated. For let t_0 be a fixed real number such that $0 < t_0 \leqslant r$ and let t be another such number. If $t < t_0$, then we have $C_t \subset P_t \subset X \setminus C_{t_0}$ ($P_t \subset P_{t_0}$ and $P_t \cap C_{t_0} = \emptyset$). Since P_t is connected, it is clear that for any realization of of $X \setminus C_{t_0}$ as the union of open sets M_1 and M_2 where $M_1 \cap M_2 = \emptyset$, we must have $P_t \subset M_1$ or $P_t \subset M_2$. Thus either $C_t \subset M_1$ or $C_t \subset M_2$. Now suppose that $t_0 < t$. Then $P_{t_0} \subset P_t$ and so $X \setminus P_t \subset X \setminus P_{t_0}$. This implies that $Q_t \subset X \setminus P_{t_0}$. Since $C_t \cap P_{t_0} = \emptyset$, we also have $C_t \subset X \setminus P_{t_0}$. Thus

$$C_t \subset Q_t \cup C_t \subset X \setminus P_{t_0} \subset X \setminus C_{t_0}. \tag{3}$$

But $Q_t \subset Q_t \cup C_t \subset \mathrm{Cl}(Q_t)$. Therefore $Q_t \cup C_t$ is connected. So C_t is contained in a connected subset of $X \setminus C_{t_0}$. Now we can conclude, as above, that if M_1 and M_2 are open subsets of $X \setminus C_{t_0}$ such that $M_1 \cup M_2 = X \setminus C_{t_0}$ and $M_1 \cap M_2 = \emptyset$, then either $C_t \subset M_1$ or $C_t \subset M_2$. This establishes our second claim.

Thus $\{C_t: 0 < t \leqslant r\}$ is an uncountable, disjoint, non-separated family of cuttings of X. Certainly $q \notin \bigcup \{C_t: 0 < t \leqslant r\}$. Theorem 2 then implies that there are C_{t_1} and C_{t_2} in $\{C_t: 0 < t \leqslant r\}$ such that C_{t_1} separates $\{q\}$ and C_{t_2}. First observe that we cannot have $t_1 < t_2$. For if this were the case we would have, as in (3) above, $C_{t_2} \subset Q_{t_2} \cup C_{t_2} \subset X \cup C_{t_1}$. But also $\{q\} \subset Q_{t_2} \cup C_{t_2}$. Thus $C_{t_2} \cup \{q\} \subset Q_{t_2} \cup C_{t_2}$, a connected set in the complement of C_{t_1}. So we must have $t_2 < t_1$. Now let M_1 and M_2 be open subsests of $X \setminus C_{t_1}$ such that $M_1 \cup M_2 = X \setminus C_{t_1}$, $M_1 \cap M_2 = \emptyset$, $\{q\} \subset M_1$ and $C_{t_2} \subset M_2$. Since $t_2 < t_1$, we have $P_{t_2} \subset X \setminus C_{t_1}$. Thus, since P_{t_2} is connected and $P_{t_2} \cap M_2 \neq \emptyset$ ($C_{t_2} \subset P_{t_2}$), we must have $P_{t_2} \subset M_2$. But $p \in P_{t_2}$. Therefore C_{t_1} separates $\{p\}$ and $\{q\}$, contradicting the fact that none of the C_t's separate $\{p\}$ and $\{q\}$. This contradiction establishes the theorem.

In [2] (see also [1]) the following question is raised:

Suppose X is a continuum with the property that every open connected subset of X is a semi-continuum. Does it follow that X is locally connected?

Obviously every continuum is a generalized continuum. Therefore, we obtain an affirmative answer to this question as an immediate corollary to theorem 3:

COROLLARY 1. *If X is a continuum with the property that every open connected subset of X is a semi-continuum, then X is locally connected.*

It is perhaps interesting to note that this corollary fails in the non-metric setting. Below we give an example of compact connected Hausdorff space with the property that each of its open connected subsets is arcwise connected, but which fails to be locally connected.

The underlying set of our example X consists of the 2-simplex in E^2 (real Euclidean 2-space) bounded by the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$. We topologize X as follows:

If $(x, y) \in X$ and $x \neq 0$, then (x, y) lies on a segment joining two points of the form $(0, c)$ and $(c, 0)$ where $c \neq 0$. A basic neighborhood of (x, y) then consists of an open subinterval of this segment which contains (x, y) and does not contain $(0, c)$. Now suppose we have some $(0, y) \in X$. Let $\varepsilon > 0$ be given and for $i = 1, 2, \ldots, n$ (n some positive integer) let F_i denote a closed subset of the segment joining some two points $(c_i, 0)$ and $(0, c_i)$ where $c_i \neq 0$. Then $F_1, F_2, \ldots, F_n$ and ε determine the basic neighborhood $\{(z, w) \in X: |y-w| < \varepsilon\} \setminus (F_1 \cup F_2 \cup \ldots \cup F_n)$ of $(0, y)$. It is not difficult to verify that X has the properties alleged above.

REFERENCES

[1] S. Bergman, R. Duda, B. Knaster, Jan Mycielski et A. Schinzel, *Kazimierz Zarankiewicz, 2. V. 1902 — 5. IX. 1959*, Colloquium Mathematicum 12 (1964), p. 277-288.

[2] B. Knaster, *Sur les notions de coupure et de séparation dans les continus non péaniens*, Annales Polonici Mathematici 16 (1937), p. 197.

[3] G. T. Whyburn, *Analytic topology*, American Mathematical Society Colloquium Publications 28, revised printing, 1955.

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