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# *INITIAL TOPOLOGIES*

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In this note some theorems on product topologies will be generalized to apply to initial topologies.

For each $j$ in an index set $I$ let $S_j$ be a topological space and let $f_j$ be a function on a set $R$ into $S_j$. Then by the *initial topology* for $R$ determined by the family $(S_j, f_j)_{j\in I}$ is meant the smallest topology which renders each $f_j$ continuous on $R$.

The following two theorems on initial topologies are from Bourbaki (1):

THEOREM 1. *A filterbase $\mathfrak{B}$ in a space $R$ with the initial topology determined by the family $(S_j, f_j)_{j\in I}$ converges to a point $x$ of $R$ if and only if for each $j \in I$ the filterbase $f_j(\mathfrak{B})$ converges to the point $f_j(x)$.*

THEOREM 2. *Let $f$ be a function on a topological space $Q$ into a space $R$ with the initial topology determined by the family $(S_j, f_j)_{j\in I}$. Then $f$ is continuous if and only if $f_j \circ f$ is continuous for each $j \in I$.*

The set $R$ is said to be *big enough* for the family $(S_j, f_j)_{j\in I}$ if and only if for each $j \in I$ and each $y \in S_j$ there is an element $x$ of $R$ such that $f_j(x) = y$. Thus $R$ is big enough for $(S_j, f_j)_{j\in I}$ if and only if the evaluation map $x \to \big(f_j(x)\big)_{j\in I}$ of $R$ into $\prod\limits_{j\in I} S_j$ is surjective.

THEOREM 3. *Let $R$ be a topological space with the initial topology determined by a family $(S_j, f_j)_{j\in I}$. If $R$ is big enough for the family $(S_j, f_j)_{j\in I}$, then for any $k \in I$ and $a$ in $R$ there is a subspace of $R$ containing $a$ and homeomorphic with $S_k$.*

Proof. We define a function $g$ on $S_k$ into $R$ by setting $g\big(f_k(a)\big) = a$ and choosing for any other $y$ of $S_k$ a unique element $g(y)$ in $R$ such that $f_k\big(g(y)\big) = y$ and $f_j\big(g(y)\big) = f_j(a)$ for every $j \in I \setminus \{k\}$ (this choice is possible, since $R$ is big enough). Let $f_k$ be the restriction of $f_k$ to $g(S_k)$. Then $a$ belongs to $g(S_k)$ and $f_k\big(g(y)\big) = y$ for all $y$ in $S_k$, and $g\big(f_k(x)\big) = x$ for all $x$ in $g(S_k)$. Consequently, $g$ is 1-1 on $S_k$ and $f_k$ is its inverse. Moreover, $g$ is also continuous on $S_k$; in fact, $f_k\big(g(y)\big) = y$ for all $y$ in $S_k$

(1) N. Bourbaki, *Topologie générale*, Paris 1961.

and so $f_k \circ g$ is continuous as the identity mapping on $S_k$; for all $j \in I \setminus \{k\}$, and all $y$ in $S_k$ we have $f_j(g(y)) = f_j(a)$, thus $f_j \circ g$ is also continuous as a constant function on $S_k$. This completes the proof by Theorem 2, because the inverse of $g$ is the restriction of a continuous function.

For each $j \in I$ let $f_j$ be a function on a set $R$ into a set $S_j$. The family $(f_j)_{j \in I}$ is said to *distinguish points* of $R$ if and only if for any two distinct points $x$ and $y$ of $R$ there is a $j \in I$ such that $f_j(x) \neq f_j(y)$. Thus the family $(f_j)_{j \in I}$ distinguishes points of $R$ if and only if the evaluation map $x \to (f_j(x))_{j \in I}$ of $R$ into $\prod_{j=1} S_j$ is injective.

From this definition and Theorem 3 easily follows

COROLLARY 4. *Let $R$ be a topological space with the initial topology determined by the family $(S_j, f_j)_{j \in I}$. Then $R$ is separated if each $S_j$ is separated and the family $(f_j)_{j \in I}$ distinguishes points of $R$. Conversely, if $R$ is separated (respectively regular or completely regular) and big enough for the family $(S_j, f_j)_{j \in I}$, then each $S_j$ is separated (respectively regular or completely regular).*

THEOREM 5. *Let $R$ be a space with the initial topology determined by the family $(S_j, f_j)_{j \in I}$. Then $R$ is regular (respectively completely regular) if, for each $j \in I$, the space $S_j$ is regular (respectively completely regular).*

Proof. Let $x$ be any point of $R$ and let $A$ be a member of a local base at $x$. Then $A$ contains an open set $U$ of the form $\bigcap_{j \in H} f_j^{-1}(U_j)$, where $H$ is a finite subset of $I$ and $U_j$ is for each $j \in H$ an open subset of $S_j$ and $x \in U$. Since $S_j$ is regular, there is a closed neighbourhood $V_j$ of $f_j(x)$ in $S_j$ such that $V_j \subseteq U_j$. But $f_j$ is continuous, thus $f_j^{-1}(V_j)$ is a closed neighbourhood of $x$ and $f_j^{-1}(V_j) \subseteq f_j^{-1}(U_j)$ for each $j \in H$. Hence $\bigcap_{j \in H} f_j^{-1}(V_j)$ is a closed neighbourhood of $x$ and it is contained in $U$, thus $R$ is regular.

To prove that $R$ is completely regular it suffices to consider any member $W$ of a base of initial topology containing $x$ and being of the form $f_j^{-1}(W_j)$, where $W_j$ is an open subset of $S_j$ for some $j \in I$. Since $S_j$ is completely regular there is a continuous function $h$ on $S_j$ into $[0, 1]$ such that $h(f_j(x)) = 0$ and $h(z) = 1$ for all $z \in S_j \setminus W_j$. Then the function $h \circ f_j$ on $R$ into $[0, 1]$ is continuous, $h(f_j(x)) = 0$ and $h(f_j(y)) = 1$ for all $y \in R \setminus W$. Thus $R$ is completely regular.

THEOREM 6. *Let $R$ be a space with the initial topology determined by the family $(S_j, f_j)_{j \in J}$, and big enough for this family. Then $R$ is compact (respectively connected) if and only if for each $j \in I$ the space $S_j$ is compact (respectively connected).*

Proof. Let $\mathfrak{A}$ be an ultrafilter on $R$, then for each $j \in I$, $f_j(\mathfrak{u})$ is a base of an ultrafilter in $S_j$ and converges to a point $x_j$ of $S_j$, since $S_j$ is compact. Because $R$ is big enough, there is an element $x$ in $R$ such that $f_j(x) = x_j$. The ultrafilter $\mathfrak{A}$ clearly converges to $x$. Thus $R$ is compact.

Let $A$ and $B$ be two non-void open sets such that $R = A \cup B$. To prove that $R$ is connected, it suffices to show that $A \cap B$ is non-void. Since they are open, $A$ and $B$ respectively contain two non-void members $\bigcap_{j\in G} f_j^{-1}(A_j)$ and $\bigcap_{j\in H} f_j^{-1}(B_j)$ of a base such that $G$ and $H$ are two finite subsets of $I$, $A_j$ is a non-void open subset of $S_j$ for $j \in G$ and $B_j$ is non-void open subset of $S_j$ for $j \in H$. Pick up an element $a$ from $\bigcap_{j\in G} f_j^{-1}(A_j)$ and an element $b$ from $\bigcap_{j\in H} f_j^{-1}(B_j)$ such that $f_j(a) = f_j(b)$ for all $j \in I \setminus (G \cup H)$ (this choice is possible, because $R$ is big enough).

Let $G$ be the set $\{j_1, \ldots, j_m\}$ and $H \setminus G$ the set $\{j_{m+1}, \ldots, j_{m+n}\}$, where $m$ and $n$ are non-negative integers. Now for each $p (1 \leqslant p \leqslant m+n)$ we define a subset $C_p$ of $R$ as follows: $C_1$ is the set of all $x \in R$ such that $f_j(x) = f_j(a)$ for all $j \in I$ other than $j_1$; for $p$ such that $1 < p \leqslant m+n$ let $C_p$ be the set of all $x \in R$ such that $f_{j_q}(x) = f_{j_q}(b)$, $q = 1, \ldots, (p-1)$ and $f_j(x) = f_j(a)$ for all $j \in I \setminus \{j_1, \ldots, j_p\}$. For all $p = 1, \ldots, m+n$, by Theorem 3, $C_p$ is homeomorphic with the connected space $S_{j_p}$ and hence itself a connected subspace of $R$. Clearly, $a \in C_1$ and $b \in C_{m+n}$. Moreover, for each $p = 1, \ldots, m+n-1$, $C_p$ and $C_{p+1}$ have a common element $x$ such that $f_{j_q}(x) = f_{j_q}(b)$ for $q = 1, \ldots, p$ and $f_j(x) = f_j(a)$ for $j \in I \setminus \{j_1, \ldots, j_p\}$. Consequently, the set $C = \bigcup_{1 \leqslant p \leqslant m+n} C_p$ is connected. Now $C \subseteq A \cup B$ and as $C \cap A$ and $C \cap B$ contain $a$ and $b$ respectively, then $C \cap A \cap B$ is non-void. So $A \cap B$ is non-void. Thus $R$ is connected.

If $R$ is compact (respectively connected), then for each $j \in I$ the space $S_j$ is compact (respectively connected), since $f_j$ is continuous and $f_j(R) = S_j$. This completes the proof.

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