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SOME OPERATIONS RELATED WITH TRANSLATION

BY

WITOLD SEREDYŃSKI (WROCŁAW)

0. Introduction. Given an ideal $\mathfrak{I}$ of subsets of the real line $\boldsymbol{R}$, let $\mathfrak{I}^*$ be the family of all subsets A of $\boldsymbol{R}$ such that for every $K \in \mathfrak{I}$ there exists $a \in \boldsymbol{R}$ with $(A+a) \cap K = \emptyset$. For $\mathfrak{I}$ being the ideal $\mathfrak{K}$ of first category subsets of $\boldsymbol{R}$, Galvin et al. showed in [4] that $\mathfrak{K}^*$ is the ideal of strong measure zero subsets of the real line. Carlson [2] examined the family $\mathfrak{L}^*$, where $\mathfrak{L}$ is the ideal of Lebesgue negligible subsets of $\boldsymbol{R}$. In particular, he proved that it is relatively consistent that $\mathfrak{L}^*$ is the ideal of countable subsets of the reals, the result that corresponds to a well-known theorem of Laver [6] on the consistency of Borel's Conjecture. In this paper we investigate some general properties of the operation *. It will show some analogy with the operations used for the construction of Lusin or Sierpiński sets. We apply this operation to the ideals of bounded subsets of the real line and to the ideal of subsets of the real line with finite Lebesgue measure.

We will use the standard set-theoretical notation and terminology, e.g., the cardinality of a set X is denoted by $|X|$. For any cardinal number $\varkappa$ let $[X]^{<\varkappa}$ (resp. $[X]^{\leqslant\varkappa}$) denote

$$\{A \subseteq X: |A| < \varkappa\} \quad (\text{resp. } \{A \subseteq X: |A| \leqslant \varkappa\}).$$

For a given set X let $\mathscr{P}(X)$ denote the family of all subsets of X. For $\mathfrak{I}, \mathfrak{J} \subsetneq \mathscr{P}(X)$ we will consider the following cardinal numbers:

$$\operatorname{add}(\mathfrak{I}) = \min\{|\mathscr{X}|: \mathscr{X} \subseteq \mathfrak{I}, \bigcup \mathscr{X} \notin \mathfrak{I}\},$$

$$\operatorname{cov}(\mathfrak{I}) = \min\{|\mathscr{X}|: \mathscr{X} \subseteq \mathfrak{I}, \bigcup \mathscr{X} = X\} \text{ or } \infty \text{ if } \bigcup \mathfrak{I} \neq X,$$

$$\operatorname{non}(\mathfrak{I}) = \min\{|A|: A \notin \mathfrak{I}\},$$

$$\operatorname{cf}(\mathfrak{I}) = \min\{|\mathscr{X}|: \mathscr{X} \subseteq \mathfrak{I} \ \forall A \in \mathfrak{I} \ \exists B \in \mathscr{X} \ A \subseteq B\},$$

$$\operatorname{add}(\mathfrak{I}, \mathfrak{J}) = \min\{|\mathscr{X}|: \mathscr{X} \subseteq \mathfrak{J} \wedge \bigcup \mathscr{X} \notin \mathfrak{I}\}.$$

We say that $\mathscr{B}$ is a *base* of $\mathfrak{I}$ if $\mathscr{B} \subseteq \mathfrak{I}$ and for every $A \in \mathfrak{I}$ there exists $\mathscr{B} \ni B \supseteq A$. By this definition the minimal cardinality of a base of $\mathfrak{I}$ is $\operatorname{cf}(\mathfrak{I})$.

For any sets A, B, let ${}^A B$ denote the set of all functions $f: A \to B$. In particular, ${}^\omega\omega$ is the Baire space. If $f, g \in {}^\lambda\varkappa$, where λ and $\varkappa$ are cardinal numbers, then, by definition, $f \prec g$ iff there exists an ordinal $\alpha < \lambda$ such that for every $\alpha < \xi < \lambda$ we have $f(\xi) < g(\xi)$. Let

$$\mathfrak{d} = \min\{|\mathscr{X}|: \mathscr{X} \subseteq {}^{\omega}\omega \wedge \forall f \in {}^{\omega}\omega\ \exists g \in \mathscr{X}\ f \prec g\}.$$

Let $\mathfrak{L}$ denote the σ-ideal of Lebesgue negligible subsets of $\boldsymbol{R}$, and let $\mathfrak{K}$ denote the σ-ideal of meagre subsets of $\boldsymbol{R}$. Let λ (resp. λ_n) denote the Lebesgue measure on $\boldsymbol{R}$ (resp. $\boldsymbol{R}^n$), and λ^* (resp. λ_*) denote outer (resp. inner) measure on $\boldsymbol{R}$. Let

$$\mathfrak{L}_n^{<\infty} = \{X \subseteq \boldsymbol{R}^n: \lambda_n^*(X) < \infty\}.$$

We say that a subset X of $\boldsymbol{R}$ is of *strong measure zero* if for any infinite sequence of positive reals $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ there exists a sequence of intervals $I_0, I_1, I_2, \ldots$ such that

$$\operatorname{diam}(I_i) < \varepsilon_i \text{ for } i \in \omega \quad \text{and} \quad X \subseteq \bigcup_{i \in \omega} I_i.$$

Let $\mathfrak{C}$ denote the σ-ideal of sets of strong measure zero.

For subsets A, B of an abelian group $\langle G, +, 0\rangle$ let

$$A+B = \{a+b: a \in A, b \in B\}.$$

Similarly we define $A-B$, $-A$, and $-\mathfrak{I} = \{-A: A \in \mathfrak{I}\}$, where $\mathfrak{I}$ is a family of subsets of G.

If $X = G$ and $\langle G, +, 0\rangle$ is an abelian group, then we define

$$\operatorname{add}_t(\mathfrak{I}) = \min\{|A|: A \subseteq G\ \exists B \in \mathfrak{I}\ A+B \notin \mathfrak{I}\},$$

$$\operatorname{cov}_t(\mathfrak{I}) = \min\{|A|: A \subseteq G\ \exists B \in \mathfrak{I}\ A+B = G\},$$

$$\operatorname{cf}_t(\mathfrak{I}) = \min\{|\mathscr{X}|: \forall A \in \mathfrak{I}\ \exists t \in G\ \exists B \in \mathscr{X}\ A+t \subseteq B\},$$

$$\operatorname{add}_t(\mathfrak{I}, \mathfrak{J}) = \min\{|A|: \exists B \in \mathfrak{J}\ A+B \notin \mathfrak{I}\}.$$

We say that $\mathscr{B} \subseteq \mathfrak{I}$ is a *transitive base* of $\mathfrak{I}$ if for every $A \in \mathfrak{I}$ there exist $B \in \mathscr{B}$ and $t \in G$ such that $A+t \subseteq B$. By this definition the minimal cardinality of a transitive base of $\mathfrak{I}$ is $\operatorname{cf}_t(\mathfrak{I})$.

The symbol $\mathfrak{I} \perp \mathfrak{J}$ means that there are $A \in \mathfrak{I}$ and $B \in \mathfrak{J}$ such that $X = A \cup B$. It is well known that $\mathfrak{K} \perp \mathfrak{L}$. Let

$$\mathfrak{I}^c = \{A \subseteq X: X \setminus A \in \mathfrak{I}\}.$$

We will use the following notions (see [5]):

DEFINITION 0.1. Let $\langle P, \leqslant_1\rangle$ and $\langle Q, \leqslant_2\rangle$ be two posets. A pair of mappings $\langle \alpha, \beta\rangle$ is a *Galois connection* over P and Q iff $\alpha: P \to Q$, $\beta: Q \to P$ and the following two conditions hold:

(1) If $x \leqslant_1 y$, then $\alpha(x) \geqslant_2 \alpha(y)$. If $x \leqslant_2 y$, then $\beta(x) \geqslant_1 \beta(y)$.

(2) $\beta \circ \alpha(x) \geqslant_1 x$ for $x \in P$ and $\alpha \circ \beta(x) \geqslant_2 x$ for $x \in Q$.

DEFINITION 0.2. Let $\langle P, \leqslant\rangle$ be a poset. A function $\chi\colon P\to P$ is a *closure operator* iff the following three conditions hold:

(1) If $x\leqslant y$, then $\chi(x)\leqslant\chi(y)$.

(2) $x\leqslant\chi(x)$ for each $x\in P$.

(3) $\chi\circ\chi=\chi$.

We will use only Galois connections of the form $\langle\alpha,\alpha\rangle$ over $\langle P,\leqslant_1\rangle=\langle Q,\leqslant_2\rangle$. It is easy to verify that $\alpha\circ\alpha$ is then a closure operator and $\alpha\circ\alpha\circ\alpha=\alpha$.

1. Some examples of Galois connections. Let X be a nonempty set. Let

$$\bar{\mathscr{P}}(X)=\{\mathfrak{I}\subseteq\mathscr{P}(X)\colon \forall A\in\mathfrak{I}\ \forall B\subseteq A\ \ B\in\mathfrak{I}\}.$$

Notice that if $\mathfrak{I}\in\bar{\mathscr{P}}(X)$, then

$$\operatorname{add}(\mathfrak{I})\leqslant\operatorname{cov}(\mathfrak{I}),\qquad \operatorname{non}(\mathfrak{I})\leqslant\operatorname{cf}(\mathfrak{I}),\qquad \operatorname{add}(\mathfrak{I},\mathfrak{I})=\operatorname{add}(\mathfrak{I}),$$
$$\operatorname{add}(\mathscr{P}(X)\backslash\{X\},\mathfrak{I})=\operatorname{cov}(\mathfrak{I}),$$

$\operatorname{add}(\mathfrak{I})$ is a finite or regular cardinal; if $\mathfrak{I}$ contains points, then $\operatorname{add}(\mathfrak{I})\leqslant\operatorname{non}(\mathfrak{I})$. For more on the cardinals $\operatorname{add}(\mathfrak{I})$, $\operatorname{non}(\mathfrak{I})$, $\operatorname{cf}(\mathfrak{I})$ for $\mathfrak{I}=\mathfrak{K}$ or $\mathfrak{L}$ see [3].

Define the function $\mathscr{F}\colon \bar{\mathscr{P}}(X)\times\bar{\mathscr{P}}(X)\to\bar{\mathscr{P}}(X)$ by

$$\mathscr{F}(\mathfrak{I},\mathfrak{J})=\{A\subseteq X\colon \forall B\in\mathfrak{J}\ A\cap B\in\mathfrak{I}\}.$$

For a given $\mathfrak{I}\in\bar{\mathscr{P}}(X)$ we define the function $\mathscr{F}^{\mathfrak{I}}\colon\bar{\mathscr{P}}(X)\to\bar{\mathscr{P}}(X)$ by

$$\mathscr{F}^{\mathfrak{I}}(J)=\mathscr{F}(\mathfrak{I},\mathfrak{J}).$$

PROPOSITION 1.1. *If $\mathfrak{I}\in\bar{\mathscr{P}}(X)$, then the pair $\langle\mathscr{F}^{\mathfrak{I}},\mathscr{F}^{\mathfrak{I}}\rangle$ is a Galois connection over $\langle\bar{\mathscr{P}}(X),\subseteq\rangle$. Moreover, if $\mathfrak{J}_1,\mathfrak{J}_2\in\bar{\mathscr{P}}(X)$, $\mathfrak{J}_1\perp\mathfrak{J}_2$, $\mathfrak{J}_1$ is an ideal and $\mathfrak{I}\subseteq\mathfrak{J}_1$, then $\mathscr{F}^{\mathfrak{I}}(\mathfrak{J}_2)\subseteq\mathfrak{J}_1$.*

Proof. The proof of property (1) of Definition 0.1 is obvious. To see (2) we claim that

$$\forall A\in\mathfrak{J}\ \forall B\in\mathscr{F}^{\mathfrak{I}}(\mathfrak{J})\ A\cap B\in\mathfrak{I}.$$

Let $A\in\mathfrak{J}$, $B\in\mathscr{F}^{\mathfrak{I}}(\mathfrak{J})$. Then for every $C\in\mathfrak{J}$ we have $B\cap C\in\mathfrak{I}$; hence, in particular, $B\cap A\in\mathfrak{I}$. So the claim is true, and $\mathfrak{J}\subseteq\mathscr{F}^{\mathfrak{I}}(\mathscr{F}^{\mathfrak{I}}(\mathfrak{J}))$ for every $\mathfrak{J}$.

Suppose now that

$$X=A\cup B,\quad A\in\mathfrak{J}_1,\ B\in\mathfrak{J}_2 \text{ and } A\cap B=\emptyset.$$

If $C\in\mathscr{F}^{\mathfrak{I}}(\mathfrak{J}_2)$, then $C\cap B\in\mathfrak{I}\subseteq\mathfrak{J}_1$, so

$$C=(C\cap A)\cup(C\cap B)\in\mathfrak{J}_1.$$

Hence the proposition is proved.

For a fixed cardinal $\varkappa$ let $\pounds_\varkappa=\mathscr{F}^{[\mathbf{R}]^{<\varkappa}}$. It is clear that $\pounds_{\omega_1}(\mathfrak{K})$ is the ideal generated by Lusin sets. By 1.1 the operation $\pounds_\varkappa$ has the property

$$\pounds_\varkappa\circ\pounds_\varkappa\circ\pounds_\varkappa=\pounds_\varkappa.$$

The second part of 1.1 applied to $\mathfrak{L}_{\omega_1}$ and ideals $\mathfrak{K}$ and $\mathfrak{L}$ states that Lusin sets are negligible and Sierpiński sets are meagre.

Notice that the notions introduced above have transitive counterparts in abelian groups, e.g., transitive counterparts of add, cov, cf are add_t, cov_t, cf_t, respectively. There exists also a transitive version of the function $\mathscr{F}$.

Let $\langle G, +, 0\rangle$ be an abelian group. Let

$$\tau(G) = \{\mathfrak{I} \in \bar{\mathscr{P}}(G)\colon\ \forall A \in \mathfrak{I}\ \ \forall t \in G\ \ A+t \in G\}.$$

Notice that if $\mathfrak{I}$, $\mathfrak{J} \in \tau(X)$, then

$$\mathrm{add}(\mathfrak{I}) \leqslant \mathrm{add}_t(\mathfrak{I}), \quad \mathrm{add}(\mathfrak{I}, \mathfrak{J}) \leqslant \mathrm{add}_t(\mathfrak{I}, \mathfrak{J}),$$

$$\mathrm{cov}(\mathfrak{I}) \leqslant \mathrm{cov}_t(\mathfrak{I}), \quad \mathrm{cf}_t(\mathfrak{I}) \leqslant \mathrm{cf}(\mathfrak{I}) \quad \mathrm{add}_t(\mathfrak{I}, \mathfrak{I}) = \mathrm{add}_t(\mathfrak{I}),$$

$$\mathrm{add}_t(\mathscr{P}(G)\backslash\{G\}, \mathfrak{I}) = \mathrm{cov}_t(\mathfrak{I}).$$

Define the function $\mathscr{F}_t\colon \tau(G)\times\tau(G)\to\tau(G)$ by

$$\mathscr{F}_t(\mathfrak{I}, \mathfrak{J}) = \{A \subseteq G\colon\ \forall B \in \mathfrak{J}\ \ \exists t \in G\ \ (A+t)\cap B \in \mathfrak{I}\}.$$

Analogously we define $\mathscr{F}_t^{\mathfrak{I}}$ for $\mathfrak{I} \in \tau(G)$.

PROPOSITION 1.1′. *If $\mathfrak{I} \in \tau(G)$, then the pair $(\mathscr{F}_t^{\mathfrak{I}}, \mathscr{F}_t^{\mathfrak{I}})$ is a Galois connection over $\langle \tau(G), \subseteq\rangle$. Moreover, if $\mathfrak{J}_1, \mathfrak{J}_2 \in \tau(G)$, $\mathfrak{J}_1 \perp \mathfrak{J}_2$, $\mathfrak{J}_1$ is an ideal and $\mathfrak{I} \subseteq \mathfrak{J}_1$, then $\mathscr{F}_t^{\mathfrak{I}}(\mathfrak{J}_2) \subseteq \mathfrak{J}_1$.*

The proof is an easy modification of the proof of Proposition 1.1.

PROPOSITION 1.2. *If $\mathfrak{I}, \mathfrak{J} \in \bar{\mathscr{P}}(X)$ (resp. $\mathfrak{I}, \mathfrak{J} \in \tau(G)$) are ideals and $\mathfrak{I} \perp \mathfrak{J}$, then*

$$\mathscr{F}(\mathfrak{I}, \mathfrak{J}) = \mathfrak{I} \quad (\textit{resp. } \mathscr{F}_t(\mathfrak{I}, \mathfrak{J}) = \mathfrak{I}).$$

In particular,

$$\mathscr{F}(\mathfrak{K}, \mathfrak{L}) = \mathscr{F}_t(\mathfrak{K}, \mathfrak{L}) = \mathfrak{K}, \qquad \mathscr{F}(\mathfrak{L}, \mathfrak{K}) = \mathscr{F}_t(\mathfrak{L}, \mathfrak{K}) = \mathfrak{L}.$$

Proof. The inclusion $\supseteq$ is trivial. Let

$$A \in \mathfrak{I}, \quad B \in \mathfrak{J}, \quad A \cup B = G \quad \text{and} \quad C \in \mathscr{F}_t(\mathfrak{I}, \mathfrak{J}).$$

Then there is $t \in G$ such that

$$C+t = \big((C+t)\cap A\big) \cup \big((C+t)\cap B\big) \in \mathfrak{I}.$$

Hence the second inclusion also holds.

Define the function $*\colon \tau(G)\to\tau(G)$ by

$$\mathfrak{I}^* = \mathscr{F}_t(\{\emptyset\}, \mathfrak{I}).$$

By Proposition 1.1′ we have $\mathfrak{I}^{***} = \mathfrak{I}^*$. Obviously,

$$\mathfrak{I}^* = \{A \subseteq G\colon\ \forall B \in \mathfrak{I}\ \ \exists t \in G\ \ (A+t)\cap B = \emptyset\}$$
$$= \{A \subseteq G\colon\ \forall B \in \mathfrak{I}\ \ A - B \neq G\}.$$

The operation * can be considered not only as a particular case of $\mathscr{F}_{\mathfrak{t}}$ but also as a case of the operation

$$\mathscr{G}_{\mathfrak{t}}\colon \tau(G)\times\tau(G)\to\tau(G)$$

connected with the second form of the above notation for $\mathfrak{J}^*$:

$$\mathscr{G}_{\mathfrak{t}}(\mathfrak{J}, \mathfrak{J}) = \{A \subseteq G\colon\ \forall B\in\mathfrak{J}\ \ A+B\in\mathfrak{J}\}.$$

Let

$$\mathscr{G}'_{\mathfrak{t}}(\mathfrak{J}, \mathfrak{J}) = \{A \subseteq G\colon\ \forall B\in\mathfrak{J}\ \ A-B\in\mathfrak{J}\} = \mathscr{G}_{\mathfrak{t}}(\mathfrak{J}, -\mathfrak{J}).$$

It is easy to see that

$$\mathfrak{J}^* = \mathscr{G}'_{\mathfrak{t}}(\mathscr{P}(G)\backslash\{G\}, \mathfrak{J}),$$

and if $\mathfrak{J} = -\mathfrak{J}$, then

$$\mathfrak{J}^* = \mathscr{G}_{\mathfrak{t}}(\mathscr{P}(G)\backslash\{G\}, \mathfrak{J}).$$

PROPOSITION 1.3. *If $\mathfrak{J}\in\tau(G)$, then the pair $(\mathscr{G}_{\mathfrak{t}}^{\mathfrak{J}}, \mathscr{G}_{\mathfrak{t}}^{\mathfrak{J}})$ is a Galois connection over $\langle\tau(G), \subseteq\rangle$. If $\mathfrak{J} = -\mathfrak{J}$, then the pair $(\mathscr{G}_{\mathfrak{t}}'^{\mathfrak{J}}, \mathscr{G}_{\mathfrak{t}}'^{\mathfrak{J}})$ is also a Galois connection.*

Proof. We omit the proof of property (1) of a Galois connection. In order to prove the second property we claim that

$$\forall A\in\mathfrak{J}\ \forall B\in\mathscr{G}_{\mathfrak{t}}^{\mathfrak{J}}(\mathfrak{J})\ \ A+B\in\mathfrak{J}.$$

Let $A\in\mathfrak{J}$ and $B\in\mathscr{G}_{\mathfrak{t}}^{\mathfrak{J}}(\mathfrak{J})$. Then $B+A\in\mathfrak{J}$ for any $C\in\mathfrak{J}$; hence $B+A\in\mathfrak{J}$.

The proof of the second statement is analogous.

Define the function $g\colon \tau(G)\to\tau(G)$ by

$$g(\mathfrak{J}) = \mathscr{G}_{\mathfrak{t}}(\mathfrak{J}, \mathfrak{J}) = \{A \subseteq G\colon\ \forall B\in\mathfrak{J}\ \ A+B\in\mathfrak{J}\}.$$

The operations $\pounds_\varkappa$ and * are the projections of $\mathscr{F}$ (resp. $\mathscr{F}_{\mathfrak{t}}$ or $\mathscr{G}_{\mathfrak{t}}$), and by 1.1, 1.1′, 1.3 they are Galois connections. The operation g is the diagonal of $\mathscr{G}_{\mathfrak{t}}$ and, unfortunately, in general it is neither an operation of closure, interior nor a Galois connection. (See the remark below Corollary 5.6.)

PROPOSITION 1.4. *For any $\mathfrak{J}\in\tau(G)$ we have $g(\mathfrak{J})\subseteq\mathfrak{J}$, and if $\mathfrak{J} = -\mathfrak{J}$, then $g(\mathfrak{J})\subseteq\mathfrak{J}\cap\mathfrak{J}^*$. Moreover, $g\circ g = g$.*

Proof. It is sufficient to prove that

$$g(\mathfrak{J}) \subseteq g(g(\mathfrak{J})).$$

Let $A\in g(\mathfrak{J})$. We claim that

$$\forall B\in g(\mathfrak{J})\ \ A+B\in g(\mathfrak{J}).$$

Let $B\in g(\mathfrak{J})$. For every $D\in\mathfrak{J}$, $B+D\in\mathfrak{J}$. Since $A\in g(\mathfrak{J})$, we have

$$A+(B+D)\in\mathfrak{J},$$

so $(A+B)+D\in\mathfrak{J}$ for every $D\in\mathfrak{J}$. Therefore $A+B\in g(\mathfrak{J})$.

Notice that the Galvin–Mycielski–Solovay theorem [4] may be formulated in our notation as follows:

$$\mathfrak{K}^* = \mathfrak{C}.$$

Recall that Rothberger in [8] proved that $\operatorname{cov}(\mathfrak{K}) \leqslant \operatorname{non}(\mathfrak{C})$. The following proposition is a generalization of this inequality:

PROPOSITION 1.5. *If* $\mathfrak{I}$, $\mathfrak{J} \in \tau(G)$, *then*

$$\operatorname{add}_t(\mathfrak{I}, \mathfrak{J}) \leqslant \operatorname{non}\mathscr{G}_t(\mathfrak{I}, \mathfrak{J}).$$

In particular,

$$\operatorname{add}_t(\mathfrak{I}) \leqslant \operatorname{non}(g(\mathfrak{I})) \quad \textit{and} \quad \operatorname{cov}_t(\mathfrak{I}) \leqslant \operatorname{non}(\mathfrak{I}^*).$$

Rothberger's inequality is a consequence of the last inequality for $\mathfrak{I} = \mathfrak{K}$ and of the Galvin–Mycielski–Solovay theorem.

Sierpiński proved in [9] that $\pounds_{\omega_1}(\mathfrak{K}) \subseteq \mathfrak{C}$. In fact, his proof together with the Galvin–Mycielski–Solovay theorem gives a stronger result.

PROPOSITION 1.6. $\mathscr{F}_t([\boldsymbol{R}]^{\leqslant\omega}, \mathfrak{K}) = \mathscr{F}_t(\{\emptyset\}, \mathfrak{K})$.

Proof. Let $A \in \mathscr{F}_t([\boldsymbol{R}]^{\leqslant\omega}, \mathfrak{K})$. We claim that $A \in \mathfrak{C}$. Let $\varepsilon_i > 0$ for $i \in \omega$. Let $\{q_i\colon i < \omega\}$ be the set of rational numbers. Let $q_i \in I_{2i}$, where I_{2i} is an open interval such that $\lambda(I_{2i}) = \varepsilon_{2i}$. The set

$$X = \boldsymbol{R} \setminus \bigcup_{i \in \omega} I_{2i}$$

is meagre. By assumption there exists $t \in \boldsymbol{R}$ such that

$$|(A+t) \cap X| \leqslant \omega.$$

Let $(A+t) \cap X = \{r_i\colon i < \omega\}$, $r_i \in I_{2i+1}$ be intervals, and $\lambda(I_{2i+1}) = \varepsilon_{2i+1}$. Of course,

$$A+t \subseteq \bigcup_{i \in \omega} I_i,$$

so $A+t \in \mathfrak{C}$ and also $A \in \mathfrak{C}$.

Notice that the following inclusions hold:

$$\pounds_{\omega_1}(\mathfrak{K}) \subseteq \mathfrak{L}, \quad \pounds_{\omega_1}(\mathfrak{L}) \subseteq \mathfrak{K}, \quad \mathfrak{C} \subseteq \mathfrak{L}, \quad \mathfrak{L}^* \subseteq \mathfrak{K}$$

(see 1.1 and 1.1′).

2. The cardinality of sets in $\mathscr{F}_t(\mathfrak{I}, \mathfrak{J})$. The classical theorem of Sierpiński on the existence of Lusin and Sierpiński sets can be generalized as follows:

PROPOSITION 2.1. *If* $\mathfrak{I}$, $\mathfrak{J} \in \mathscr{P}(X)$ *and*

$$\operatorname{cf}(\mathfrak{J}) = \operatorname{cov}(\mathfrak{J}) = \varkappa \leqslant \operatorname{non}(\mathfrak{I}),$$

then there exists $A \in \mathscr{F}(\mathfrak{I}, \mathfrak{J})$ *such that* $|A| = \varkappa$.

We omit an easy proof of Proposition 2.1 (cf. the proof of 2.1′ below).

By the Galvin–Mycielski–Solovay theorem, $\mathfrak{K}^*$ is a σ-ideal. T. Carlson asked if $\mathfrak{L}^*$ is an ideal. Let BC (the Borel conjecture) mean "$\mathfrak{C} = [\boldsymbol{R}]^{\leqslant\omega}$". Laver proved in [6] the consistency of BC, and Carlson proved in [2] the consistency of "$\mathfrak{L}^* = [\boldsymbol{R}]^{\leqslant\omega}$". But under some natural set-theoretical assumptions (e.g., CH, i.e., the continuum hypothesis) there are uncountable sets in $\mathfrak{C}$ (see [1]) and in $\mathfrak{L}^*$. The following proposition is a common generalization of these facts and is a transitive counterpart of 2.1.

PROPOSITION 2.1′. *If* $\mathfrak{I}, \mathfrak{J} \in \tau(G)$ *and*

$$\operatorname{cf}(\mathfrak{J}) = \operatorname{cov}(\mathfrak{J}) = \varkappa \geqslant \omega,$$

then there exists $X \in \mathscr{F}_t(\mathfrak{I}, \mathfrak{J})$ *such that* $|X| = \varkappa$.

Proof. Let $\{B_\xi\colon \xi < \varkappa\} \subseteq \mathfrak{J}$ be a base of $\mathfrak{J}$, i.e., for each $A \in \mathfrak{J}$ there is $\xi < \varkappa$ such that $A \subseteq B_\xi$. Define two sequences

$$\{t_\xi\colon \xi < \varkappa\} \subseteq G \quad \text{and} \quad \{Q_\xi\colon \xi < \varkappa\} \subseteq \mathscr{P}(G)$$

such that if $\xi \in \zeta$, then $Q_\xi \subsetneq Q_\zeta$, $|Q_\xi| \leqslant |\xi|$, and $(B_\xi + t_\xi) \cap Q_\xi \in \mathfrak{I}$. Set $Q_0 = \emptyset$ and $t_0 = 0$. For limit ordinals $\gamma < \varkappa$ let

$$Q_\gamma = \bigcup_{\xi<\gamma} Q_\xi.$$

Obviously, $|Q_\gamma| \leqslant \sup|\xi| \leqslant |\gamma|$. By 1.5,

$$\varkappa = \operatorname{cov}(\mathfrak{J}) \leqslant \operatorname{cov}_t(\mathfrak{J}) \leqslant \operatorname{non}(\mathfrak{J}^*) \leqslant \operatorname{non}(\mathscr{F}_t(\mathfrak{I}, \mathfrak{J})),$$

so $Q_\gamma \in \mathscr{F}_t(\mathfrak{I}, \mathfrak{J})$. Hence there exists t_γ such that

$$(B_\gamma + t_\gamma) \cap Q_\gamma \in \mathfrak{I}.$$

Suppose that Q_ξ is defined. Let

$$q \notin Q_\xi \cup \bigcup_{\eta\leqslant\xi} (B_\eta + t_\eta).$$

Such a q exists by $\operatorname{cov}(\mathfrak{J}) = \varkappa$. Let $Q_{\xi+1} = Q_\xi \cup \{q\}$. Obviously, $Q_{\xi+1} \in \mathscr{F}_t(\mathfrak{I}, \mathfrak{J})$. Let $t_{\xi+1}$ be such that $(B_{\xi+1} + t_{\xi+1}) \cap Q_{\xi+1} \in \mathfrak{I}$. Let

$$X = \bigcup_{\xi<\varkappa} Q_\xi.$$

Of course, $|X| = \varkappa$. We claim that $X \in \mathscr{F}_t(\mathfrak{I}, \mathfrak{J})$. It suffices to see that for every $\xi < \varkappa$ we have $(B_\xi + t_\xi) \cap X \in \mathfrak{I}$. If $q \in X \setminus Q_\xi$, then $q \notin B_\xi + t_\xi$, so

$$(B_\xi + t_\xi) \cap X = (B_\xi + t_\xi) \cap Q_\xi \in \mathfrak{I}.$$

COROLLARY 2.2 (J. Cichoń). (CH) *There exists* $X \in \mathfrak{L}^*$ *such that* $\lambda_*(\boldsymbol{R} \setminus X) = 0$.

Proof. Let $\{C_\xi\colon \xi < \omega_1\}$ be an enumeration of all Borel sets of positive measure. Then use an argument similar to that in the proof of 2.1′. The only difference is the requirement

$$q \in C_\xi \setminus \Big(Q_\xi \cup \bigcup_{\eta\leqslant\xi} (B_\eta + t_\eta)\Big).$$

COROLLARY 2.3. $([R]^{\leqslant\omega})^* \supsetneq \mathfrak{K} \cup \mathfrak{L} \cup [R]^{<2^\omega}$.

Proof. We ought to prove the existence of

$$X \in ([R]^{\leqslant\omega})^* \setminus (\mathfrak{K} \cup \mathfrak{L} \cup [R]^{<2^\omega}).$$

Let $\{C_\xi\colon \xi < 2^\omega\}$ be an enumeration of all Borel sets either of second Baire category or of positive measure. Again an argument similar to that in 2.1′, with the modification

$$q \in C_\xi \setminus (Q_\xi \cup \bigcup_{\eta \leqslant \xi} (B_\eta + t_\eta)),$$

gives the desired result.

3. Base and transitive base. Pawlikowski proved in [7] that

$$\mathrm{cf}_t(\mathfrak{L}) = \mathrm{cf}(\mathfrak{L}) \quad \text{and} \quad \mathrm{cf}_t(\mathfrak{K}) = \mathfrak{d}.$$

The following lemma was proved by F. Galvin and independently by J. Brzuchowski, J. Cichoń and B. Węglorz.

LEMMA 3.1. *Let $\mathfrak{I} \in \tau(G)$, $\mathfrak{I} = -\mathfrak{I}$ and for every $A \in \mathfrak{I}$ there exists $B \in \mathfrak{I}$ such that $A \subsetneq B$. Then $\mathrm{cf}_t(\mathfrak{I}) > 1$. If, in addition, $\mathfrak{I}$ is an ideal, then $\mathrm{cf}_t(\mathfrak{I}) \geqslant \omega$.*

Proof. Suppose not. Let $S \in \mathfrak{I}$ be such that for every $A \in \mathfrak{I}$ there exists $t \in G$ for which $A \subseteq S+t$. By assumption, $-S \in \mathfrak{I}$, so there exists $z \in S$ such that $-S \subseteq S+z$. Hence

$$S \subseteq -S-z \quad \text{and} \quad S+z \subseteq -S.$$

Therefore $-S = S+z$. Let $S \subsetneq T \in \mathfrak{I}$. There exists $x \in G$ such that $T \subseteq S+x$, so $S \subsetneq S+x$, and hence $-S \subsetneq -S-x$, i.e., $S+z \subsetneq S+z-x$. Hence $S \subsetneq S-x$, so

$$S \subsetneq S+x \subsetneq S-x+x = S.$$

Therefore, $S \subsetneq S$. This contradiction completes the proof.

The assumption $\mathfrak{I} = -\mathfrak{I}$ is essential. J. Cichoń has constructed a σ-ideal on $\boldsymbol{R}$ for which $\mathfrak{I} \neq -\mathfrak{I}$ and $\mathrm{cf}_t(\mathfrak{I}) = 1$.

THEOREM 3.2. *If $\mathfrak{I} \in \tau(\boldsymbol{R})$, $\mathfrak{I} = -\mathfrak{I}$ and $\mathfrak{I}$ is an ideal, then*

$$\mathrm{cf}(\mathfrak{I}^*) \leqslant 2^{\mathrm{cf}_t(\mathfrak{I})}.$$

Proof. Let $\mathrm{cf}_t(\mathfrak{I}) = \varkappa$. By Lemma 3.1, we have $\varkappa \geqslant \omega$. Let $\{A_\xi\colon \xi < \varkappa\}$ be a transitive base of $\mathfrak{I}$ such that $A_\xi = -A_\xi$ for $\xi < \varkappa$. Let

$$\mathscr{B} = \{\bigcap_{\xi<\varkappa} ((\boldsymbol{R}\setminus A_\xi)+t_\xi)\colon \langle t_\xi\colon \xi < \varkappa\rangle \in {}^\varkappa\boldsymbol{R}\}.$$

Of course, $|\mathscr{B}| = (2^\omega)^\varkappa = 2^\varkappa$. We claim that $\mathscr{B} \subseteq \mathfrak{I}^*$. If not, then there exists $\langle t_\xi\colon \xi < \varkappa\rangle$ such that

$$\bigcap_{\xi<\varkappa} ((\boldsymbol{R}\setminus A_\xi)+t_\xi) \notin \mathfrak{I}^*,$$

so there is $\eta < \varkappa$ such that

$$\bigcap_{\xi<\varkappa} ((\boldsymbol{R}\setminus A_\xi)+t_\xi)+A_\eta = \boldsymbol{R}.$$

In particular, $t_\eta \in (\boldsymbol{R}\setminus A_\eta)+t_\eta+A_\eta$, and hence $0 \in (\boldsymbol{R}\setminus A_\eta)+A_\eta$. This is impossible, because $A_\eta = -A_\eta$, and if $0 = t+(-t)$, then either $\{t, -t\} \subseteq \boldsymbol{R}\setminus A_\eta$ or $\{t, -t\} \subseteq A_\eta$.

We claim that $\mathscr{B}$ is a base of $\mathfrak{J}^*$. Let $C \in \mathfrak{J}^*$. Then for every $\xi < \varkappa$ there is t_ξ such that $C \cap (A_\xi+t_\xi) = \emptyset$, so $C \subseteq (\boldsymbol{R}\setminus A_\xi)+t_\xi$. Therefore

$$C \subseteq \bigcap_{\xi<\varkappa} ((\boldsymbol{R}\setminus A_\xi)+t_\xi) \in \mathscr{B}.$$

COROLLARY 3.3. (a) (J. Cichoń) $\mathrm{cf}(\mathfrak{C}) \leqslant 2^{\mathfrak{b}}$.

(b) $\mathrm{cf}(\mathfrak{L}^*) \leqslant 2^{\mathrm{cf}(\mathfrak{L})}$.

Proof. The inequalities follow from Theorem 3.2, Pawlikowski's theorem [7] and the Galvin–Mycielski–Solovay theorem.

Cichoń's original proof of the inequality (a) was different.

Let $\varkappa$, λ, δ be infinite cardinal numbers. We say that a sequence $\langle f_\xi: \xi < \delta\rangle \subseteq {}^\lambda\varkappa$ is a *scale* if $f_\xi \prec f_\zeta$ for $\xi < \zeta$ and if for every $f \in {}^\lambda\varkappa$ there is $\xi < \delta$ such that $f \prec f_\xi$. The cardinal δ is called the *length of the scale* $\langle f_\xi: \xi < \delta\rangle$. The following lemma is well known:

LEMMA 3.4. *If* CH *holds and* $2^{\omega_1} = \omega_2$, *then there exists a scale on* ${}^{\omega_1}\omega_1$ *of length* ω_2.

Proof. Let ${}^{\omega_1}\omega_1 = \{g_\alpha: \alpha < \omega_2\}$. Let $\alpha < \omega_2$. Suppose that we have a sequence

$$\langle f_\xi: \xi < \alpha\rangle \subseteq {}^{\omega_1}\omega_1$$

such that, for $\xi < \zeta < \alpha$, $f_\xi \prec f_\zeta$ and $g_\xi \prec f_\xi$ hold. Let $\{h_\eta: \eta < \omega_1\}$ be any enumeration of $\{f_\xi: \xi < \alpha\} \cup \{g_\alpha\}$. Set

$$f_\alpha(\xi) = \sup\{h_\eta(\eta): \eta < \xi\}+1.$$

It is easy to see that $f_\xi \prec f_\alpha$ for $\xi < \alpha$ and $g_\alpha \prec f_\alpha$. By the very construction, $\{f_\xi: \xi < \omega_2\}$ is the desired scale.

A special case of the following theorem was proved by J. Cichoń.

THEOREM 3.5. *Let* $\varkappa \geqslant \omega$ *be a cardinal and* λ *be a regular cardinal. Assume that there exists a scale on* ${}^\varkappa\varkappa$ *of length* λ. *Let* $\mathfrak{J} \in \bar{\mathscr{P}}(\varkappa)$ *be such that* $\mathrm{add}(\mathfrak{J}) = \mathrm{cf}(\mathfrak{J}) = \varkappa$ *and assume that for every* $A \in \mathfrak{J}$ *there exists* $B \in \mathfrak{J}$ *such that* $|B\setminus A| = \varkappa$. *Let* $\mathfrak{I} \in \bar{\mathscr{P}}(\varkappa)$ *be such that*

$$\pounds_\varkappa(\mathfrak{J}) \subseteq \mathfrak{I} \quad \text{and} \quad \mathfrak{J}^c \cap \mathfrak{I} = \emptyset.$$

Then $\mathrm{cf}(\mathfrak{I}) \geqslant \lambda$.

Proof. By the assumptions on $\mathfrak{J}$ it is easy to construct a base $\langle K_\xi: \xi < \varkappa\rangle$ of $\mathfrak{J}$ such that for $\xi < \zeta$ we have

$$K_\xi \subseteq K_\zeta, \quad |K_\zeta\setminus K_\xi| = \varkappa \quad \text{and} \quad K_0 = \emptyset.$$

Hence we may suppose that $\mathfrak{I}$ is in $\mathscr{P}(\varkappa\times\varkappa)$ and that

$$K_{\alpha+1}\setminus K_\alpha = \{\alpha\}\times\varkappa, \quad K = \alpha\times\varkappa.$$

Note that for any $f\in{}^\varkappa\varkappa$ we have

$$\{\langle\alpha,\beta\rangle\colon \beta\leqslant f(\alpha)\}\in\pounds_\varkappa(\mathfrak{I}).$$

Let $\langle f_\xi\colon \xi<\lambda\rangle$ be a scale on ${}^\varkappa\varkappa$. Suppose that

$$\mathfrak{J}\in\mathscr{P}(\varkappa),\quad \pounds_\varkappa(\mathfrak{I})\subseteq\mathfrak{J},\quad \mathfrak{I}^c\cap\mathfrak{J}=\emptyset,$$

but $\mathrm{cf}(\mathfrak{J})<\lambda$. Let $\mathscr{B}$ be a base of $\mathfrak{J}$ and $|\mathscr{B}|=\mathrm{cf}(\mathfrak{J})$. Let

$$L_\xi=\{\langle\alpha,\beta\rangle\colon \beta\leqslant f_\xi(\alpha)\}.$$

Clearly, $L_\xi\in\mathfrak{J}$. By regularity of λ there exist $\langle\xi_\eta\colon \eta<\lambda\rangle$ and $B\in\mathscr{B}$ such that $L_{\xi_\eta}\subseteq B$. Of course, $\langle f_{\xi_\eta}\colon \eta<\lambda\rangle$ is a scale as well. Moreover, $B\in\mathfrak{J}$. There exists $\{\langle\alpha_\xi,\beta_\xi\rangle\colon \xi<\varkappa\}$ such that

$$\lim\alpha_\xi=\varkappa \quad\text{and}\quad \{\langle\alpha_\xi,\beta_\xi\rangle\colon \xi<\varkappa\}\cap B=\emptyset$$

(since if not, then $[\alpha,\varkappa)\times\varkappa\subseteq B$ for any α, so $B\in\mathfrak{I}^c$, which contradicts the assumption). Let $f\in{}^\varkappa\varkappa$ be such that $f(\alpha_\xi)=\beta_\xi$ for $\xi<\varkappa$ and let $\alpha<\lambda$ be such that $f\prec f_{\xi_\alpha}$. Therefore, for a sufficiently large $\xi<\varkappa$ we have

$$\beta_\xi=f(\alpha_\xi)<f_{\xi_\alpha}(\alpha_\xi),$$

so $\langle\alpha_\xi,\beta_\xi\rangle\in L_{\xi_\alpha}\subseteq B$, which contradicts $\{\langle\alpha_\xi,\beta_\xi\rangle\colon \xi<\varkappa\}\cap B=\emptyset$.

COROLLARY 3.6 (J. Cichoń). *If* CH *holds and* $2^{\omega_1}=\omega_2$, *then*

$$\mathrm{cf}(\pounds_{\omega_1}(\mathfrak{L}))=\mathrm{cf}(\pounds_{\omega_1}(\mathfrak{K}))=\mathrm{cf}(\mathfrak{C})=\omega_2.$$

Proof. It is well known that no perfect set belongs to $\mathfrak{C}$, hence $\mathfrak{C}\cap\mathfrak{K}^c=\emptyset$. The corollary follows from 3.4, 3.5, and Sierpiński's inclusion $\pounds_{\omega_1}(\mathfrak{K})\subseteq\mathfrak{C}$.

The above corollary shows the consistency of the statement $\mathrm{cf}(\mathfrak{C})>2^\omega$.

4. **-closed families. From the equality $\mathfrak{I}^*=\mathfrak{I}^{***}$ it follows that $\mathfrak{I}\in\tau(G)$ is **-closed iff there is $\mathfrak{J}\in\tau(G)$ such that $\mathfrak{J}^*=\mathfrak{I}$. Therefore, by the Galvin–Mycielski–Solovay theorem, $\mathfrak{C}$ is a **-closed σ-ideal. It follows that if BC holds, then $[R]^{\leqslant\omega}$ is **-closed. The difficulties in describing $\mathfrak{I}^*$ and $\mathfrak{I}^{**}$ for particular families are due to the fact that the form of $\mathfrak{I}^*$ and $\mathfrak{I}^{**}$ strongly depends on set-theoretical assumptions. The following example serves as an illustration:

EXAMPLE 4.1. *Let* $X\in\mathfrak{L}$ *be a dense* G_δ*-set.*

(a) *If* BC *holds, then* $X\in\mathfrak{K}^{**}$.

(b) *If* CH *holds, then* $X\notin\mathfrak{K}^{**}$.

In particular, if BC *holds, then* $\mathfrak{K}$ *is not **-closed.*

Proof. (a) Note that $\mathfrak{L} \subseteq ([R]^{\leqslant\omega})^*$; hence if BC holds, then

$$\mathfrak{L} \subseteq \mathfrak{C}^* = \mathfrak{K}^{**}.$$

Thus (a) is proved.

(b) Let $A \in \pounds_{\omega_1}(\mathfrak{K})$, $|A| = \omega_1$. Then $A \in \mathfrak{C}$. If $X \in \mathfrak{C}^*$, then there is $x \in R$ such that $(A+x) \cap X = \emptyset$, so

$$A+x \subseteq R \backslash X \in \mathfrak{K}.$$

Hence $A \in \mathfrak{K}$, which contradicts $A \in \pounds_{\omega_1}(\mathfrak{K})$ and $|A| = \omega_1$.

We will prove that the ideal of bounded sets is $**$-closed. Let $\langle G, +, \varrho\rangle$ be an abelian metric unbounded group with invariant metric ϱ. Let $B(a, r)$ be an open ball with centre a and radius r, and let $\bar{B}(a, r)$ stand for the respective closed ball. Let

$$\mathrm{Bd} = \{X \subseteq G\colon \operatorname{diam}(X) < \infty\}.$$

THEOREM 4.2. (a) $\mathrm{Bd}^* = \{X \subseteq G\colon \forall n \in \omega\ \exists a \in G\ B(a, n) \cap X = \emptyset\}$.

(b) Bd *is* $**$*-closed.*

Proof. (a) is obvious. In order to prove (b) it suffices to show that $\mathrm{Bd}^{**} \subseteq \mathrm{Bd}$. Let $A \in \mathrm{Bd}^{**}$ and suppose that $A \notin \mathrm{Bd}$. Then there exists $\{a_n\colon n < \omega\} \subseteq A$ such that $\varrho(a_{n+1}, 0) > 3\varrho(a_n, 0)$. One may assume that $A = \{a_n\colon n < \omega\}$ and $a_0 = 0$, $a_1 \neq 0$. Let $r_i = \varrho(0, a_i)$. We put

$$X = \bigcup_n (\bar{B}(0, r_{3n+1}) \backslash B(0, r_{3n})).$$

Clearly,

$$G \backslash X = \bigcup_n (B(0, r_{3n+3}) \backslash \bar{B}(0, r_{3n+1})).$$

We claim that

$$B(a_{3n+2}, r_{3n+1}) \subseteq B(0, r_{3n+3}) \backslash \bar{B}(0, r_{3n+1}) \subseteq G \backslash X.$$

If $x \in B(a_{3n+2}, r_{3n+1})$, then

$$\varrho(x, 0) \leqslant \varrho(x, a_{3n+2}) + r_{3n+2} < r_{3n+1} + r_{3n+2} \leqslant 2r_{3n+2} < r_{3n+3},$$

so $x \in B(0, r_{3n+3})$. Suppose that $x \in \bar{B}(0, r_{3n+1})$. Then

$$r_{3n+2} \leqslant \varrho(a_{3n+2}, x) + \varrho(x, 0) < r_{3n+1} + r_{3n+1} < 3r_{3n+1},$$

which contradicts $r_{3n+2} > 3r_{3n+1}$. Hence the claim is proved. Therefore $X \in \mathrm{Bd}^*$.

Since $A \in \mathrm{Bd}^{**}$, there exists $x \in G$ such that $A + x \subseteq G \backslash X$. Let $n \in \omega$ satisfy

$$a_0 + x = x \in B(0, r_{3n+3}) \backslash \bar{B}(0, r_{3n+1}).$$

Let k be the least natural number such that

$$a_k + x \notin \bar{B}(0, r_{3n+3}).$$

From $a_k+x\in G\backslash X$ it follows that $a_k+x\notin \bar{B}(0, r_{3n+4})$. By minimality of k we have $a_{k-1}+x\in B(0, r_{3n+3})$. Therefore

$$\varrho(a_{k-1}, a_k) = \varrho(a_{k-1}+x, a_k+x)+\varrho(0, a_{k-1}+x)-\varrho(0, a_{k-1}+x)$$
$$\geqslant \varrho(0, a_k+x)-\varrho(0, a_{k-1}+x) > r_{3n+4}-r_{3n+3} \geqslant 2r_{3n+3}.$$

Note that $k\geqslant 3n+4$ (since if not, then $\varrho(a_{k-1}, a_k)\leqslant r_{k-1}+r_k\leqslant 2r_{3n+3}$, which is impossible). But by $a_{k-1}\in B(0, r_{3n+3})$ we have $r_{k-1} < r_{3n+3}$, so $k-1 < 3n+3$, which contradicts $k\geqslant 3n+4$.

5. Characterization of $(\mathfrak{L}^{<\infty})^*$. The proof of our next theorem is closely connected with the following result used by Carlson in [2]:

If $A\subseteq \boldsymbol{R}$ is bounded and infinite, then for every $\varepsilon > 0$ there exists $X\subseteq \boldsymbol{R}$ such that $\lambda(X)=\varepsilon$ and $A+X=\boldsymbol{R}$.

Let I be the interval $[0, 1)$. Let

$$\mathfrak{J}^n_k = \{A\subseteq \boldsymbol{R}^n\colon\ \forall t\in \boldsymbol{R}^n\ |A\cap(I^n+t)|\leqslant k\}\quad \text{for } k, n\in\omega.$$

Let

$$\mathfrak{J}^n = \bigcup_k \mathfrak{J}^n_k.$$

Clearly, $\mathfrak{J}^n$ is an ideal.

THEOREM 5.1.

(a) $$\{A\subseteq \boldsymbol{R}^n\colon\ \neg\,\forall\varepsilon>0\ \exists X\subseteq \boldsymbol{R}^n\ \lambda_n(X)\leqslant\varepsilon\wedge X+A=\boldsymbol{R}^n\} = \mathfrak{J}^n.$$

(b) $$(\mathfrak{L}_n^{<\infty})^* = \mathfrak{J}^n\cap \mathrm{Bd}^*.$$

For the proof we need the following

LEMMA 5.2. *For every $n\in\omega$ and $\varepsilon>0$ there exists $k\in\omega$ such that if $A\subseteq I^n$ and $|A|=k$, then there exists $X\subseteq \boldsymbol{R}^n$ for which*

$$\lambda_n(X)\leqslant\varepsilon\quad \textit{and}\quad \lambda_n(I^n\backslash(A+X))\leqslant\varepsilon.$$

Proof (D. H. Fremlin). Let $[x]$ denote the integer part of a real x. Let

$$x\pm(\mathrm{mod}\ 1)y = x\pm y-[x\pm y].$$

Let

$$X\pm(\mathrm{mod}\ 1)Y = \{x\pm(\mathrm{mod}\ 1)y\colon\ x\in X,\ y\in Y\}.$$

It suffices to prove the case $n=1$, because the case $n>1$ can be proved by an easy induction. Let $n=1$, $\varepsilon>0$. Let k be large enough to have

$$\exp(-k\varepsilon/2)<\varepsilon.$$

Since

$$\lim_{q\to\infty}(1-k/q)^{\varepsilon q/2} = \exp(-k\varepsilon/2)<\varepsilon,$$

there exists $q_0 \in \omega$ such that for every $q \geqslant q_0$ we have

$$(1-k/q)^{[\varepsilon q/2]} < \varepsilon.$$

Let $A \subseteq I$, $|A| = k$. Let $q \geqslant q_0$ be such that for every distinct $a, a' \in A$ we have

$$|a-a'| > 1/q \quad \text{and} \quad a+(1-a') > 1/q.$$

Let $I_j = [j/q, (j+1)/q)$ for $0 \leqslant j < q$. Let

$$p = [\varepsilon q/2], \quad \text{i.e.,} \quad (1-k/q)^p < \varepsilon.$$

Let ν_0 be the measure on q (identified with $\{0, 1, \dots, q-1\}$) such that, for $D \subseteq p$, $\nu_0(D) = |D|/q$. Let ν be the product measure ν_0^p on ${}^pq$. For $z = \langle z_i \colon i < p\rangle$ in ${}^pq$ let

$$F(z) = \bigcup_{i<p} I_{z_i}.$$

Then $\lambda(F(z)) \leqslant p/q \leqslant \varepsilon/2$. Let

$$G(z) = A + (\mathrm{mod}\ 1) F(z) \subseteq I.$$

We define the Borel set G by

$$G = \{(x, z) \colon x \in I, z \in {}^pq, x \notin G(z)\}.$$

Let $x \in I$. By assumption we have $|x-(\mathrm{mod}\ 1)A| = |A| = k$ and

$$|\{j \colon I_j \cap (x-(\mathrm{mod}\ 1)A) \neq \emptyset\}| = k.$$

Therefore

$$\begin{aligned}
\nu(\{z \colon (x, z) \in G\}) &= \nu(\{z \colon x \notin G(z)\}) \\
&= \nu(\{z \colon F(z) \cap (x-(\mathrm{mod}\ 1)A) = \emptyset\}) \\
&= \nu(\{z \colon I_{z_i} \cap (x-(\mathrm{mod}\ 1)A) = \emptyset \text{ for } i < p\}) \\
&= \prod_{i<p} \nu_0(\{j < q \colon I_j \cap (x-(\mathrm{mod}\ 1)A) = \emptyset\}) \\
&= \left(\frac{1}{q}|\{j < q \colon I_j \cap (x-(\mathrm{mod}\ 1)A) = \emptyset\}|\right)^p = \left(\frac{q-k}{q}\right)^p = \left(1-\frac{k}{q}\right)^p < \varepsilon.
\end{aligned}$$

Thus $(\lambda \times \nu)(G) < \varepsilon$ and, by Fubini's theorem, there exists $z \in {}^pq$ such that

$$\lambda(\{x \colon (x, z) \in G\}) < \varepsilon,$$

so

$$\lambda(I \setminus (A + (\mathrm{mod}\ 1) F(z))) < \varepsilon.$$

It is easy to see that $X = F(z) \cup (F(z)-1)$ satisfies the lemma.

Proof of the inclusions $\subseteq$ in (a) and (b).

(a) Note that Lemma 5.2 is true for r, $s \in \mathbf{Z}^n$ with the assumption $A \subseteq I^n + r$ and the assertion

$$\lambda_n(X) \leqslant \varepsilon, \quad \lambda_n((I^n+s)\setminus(A+X)) \leqslant \varepsilon.$$

Let $A \notin \mathfrak{J}^n$. Then for any $k \in \omega$ there exist $r \in \mathbf{Z}^n$ and $A_r \subseteq A$, $|A_r| = k$, $A_r \subseteq I^n + r$. Let $\varepsilon > 0$. Let

$$\mathbf{Z}^n \times \omega = \{\langle k_i, l_i \rangle : i \in \omega\}.$$

Let X_i satisfy

$$\lambda_n(X_i) \leqslant \varepsilon/2^{i+1}, \quad \lambda_n((I^n+k_i)\setminus(A_{k_i}+X_i)) \leqslant \varepsilon/2^{i+1}.$$

Let

$$X' = \bigcup_{i\in\omega} X_i.$$

Clearly,

$$\lambda_n(X') < \varepsilon \quad \text{and} \quad \mathbf{R}^n\setminus(A+X') = \bigcup_{k\in\mathbf{Z}^n} ((I^n+k)\setminus(A+X')).$$

For a fixed $k \in \mathbf{Z}^n$ we claim that

$$\lambda_n((I^n+k)\setminus(A+X')) = 0.$$

To see this let $\delta > 0$. By $|\{i: k_i = k\}| = \omega$ there is an i such that $k_i = k$ and $\varepsilon/2^{i+1} < \delta$, so

$$\lambda_n((I^n+k)\setminus(A+X')) \leqslant \lambda_n((I^n+k_i)\setminus(A_{k_i}+X_i)) \leqslant \varepsilon/2^{i+1} < \delta.$$

By a free choice of δ the claim is proved, so

$$\lambda_n((\mathbf{R}^n\setminus(A+X')) = 0.$$

Let $a_0 \in A$ and $X = X' \cup ((\mathbf{R}^n\setminus(A+X'))-a_0)$. It is easy to see that

$$\lambda_n(X) \leqslant \varepsilon \quad \text{and} \quad A+X = \mathbf{R}^n.$$

(b) By (a) we have

$$(\mathfrak{L}_n^{<\infty})^* \subseteq \mathfrak{J}^n,$$

and by $\mathrm{Bd} \subseteq \mathfrak{L}_n^{<\infty}$ we have

$$(\mathfrak{L}_n^{<\infty})^* \subseteq \mathrm{Bd}^*.$$

LEMMA 5.3. *Let*

$$\mathscr{A} = \{A \subseteq \mathbf{R}^n : (\forall a, a' \in A)(a \neq a' \to \varrho(a, a') \geqslant 1)\}.$$

If $A \in \mathfrak{J}_k^n$, *then there exist* $A_1, A_2, \ldots, A_{2^{2n}k} \in \mathscr{A}$ *such that*

$$A = \bigcup \{A_i : 1 \leqslant i \leqslant 2^{2n}k\}.$$

In particular, $\mathfrak{J}^n$ *is an ideal generated by* $\mathscr{A}$.

Proof. For any $m \in Z^n$ let

$$A \cap (I^n + m) = \{a_i^m : 1 \leqslant i \leqslant k(m)\}, \quad \text{where } k(m) \leqslant k.$$

Let

$$B_i^m = \begin{cases} \{a_i^m\} & \text{for } 1 \leqslant i \leqslant k(m), \\ \emptyset & \text{for } k(m) < i \leqslant k, \end{cases} \quad \text{for } m \in Z^n,\ 1 \leqslant i \leqslant k.$$

Let

$$B_i = \bigcup_{m \in Z^n} B_i^m.$$

Of course,

$$A = \bigcup_{i=1}^{k} B_i \quad \text{and} \quad |B_i \cap (I^n + m)| \leqslant 1 \text{ for } m \in Z^n.$$

Let

$$[0, 2)^n = \bigcup_{l=1}^{2^{2n}} I_l,$$

where I_l are cubes with the edges of length $\frac{1}{2}$. Let

$$A_i^l = B_i \cap \bigcup_{m \in Z^n} (I_l + 2m).$$

Enumerating A_i^l we obtain the required partition (because for different $m, r \in Z^n$ we have $\varrho(I_l + 2m, I_l + 2m) > 1$).

LEMMA 5.4. *Let $X \subseteq R^n$, $\lambda_n(X) < \infty$, $0 < \varepsilon < 1$. Then there exists $s(X, \varepsilon) > 0$ such that if*

$$A \subseteq R^n \setminus B(0, s(X, \varepsilon)) \quad \text{and} \quad A \in \mathscr{A},$$

then

$$\lambda_n((A + X) \cap B(0, \tfrac{1}{2})) < \varepsilon.$$

Proof. Let $s(X, \varepsilon)$ be so large that

$$\lambda_n(X \setminus B(0, s(X, \varepsilon) - 1)) < \varepsilon.$$

Let $A = \{a_i : i < \omega\}$ satisfy the assumptions. Then

$$\begin{aligned} \lambda_n((X + A) \cap B(0, \tfrac{1}{2})) &\leqslant \sum_{i \in \omega} \lambda_n((X + a_i) \cap B(0, \tfrac{1}{2})) \\ &= \sum_{i \in \omega} \lambda_n(X \cap B(-a_i, \tfrac{1}{2})) \\ &= \lambda_n\Big(\bigcup_{i \in \omega} B(-a_i, \tfrac{1}{2}) \cap X\Big) \leqslant \lambda_n(X \setminus B(0, s(X, \varepsilon) - 1)) < \varepsilon. \end{aligned}$$

LEMMA 5.5. *Let $X \in \mathfrak{L}_n^{<\infty}$, $k \in \omega$, $0 < \varepsilon < 1$. Then there exists $t(X, \varepsilon, k) > 0$ such that if*

$$A \subseteq R^n \setminus B(0, t(X, \varepsilon, k)) \quad \text{and} \quad A \in \mathfrak{J}_k^n,$$

then

$$\lambda_n((X+A)\cap B(0,\tfrac{1}{2}))<\varepsilon.$$

Proof. Let

$$t(X,\varepsilon,k)=s(X,\varepsilon/2^{2^n}k).$$

Let A satisfy the assumptions. By 5.3 there exist $A_i\in\mathscr{A}$ such that

$$A=\bigcup_{i=1}^{2^{2n}k}A_i.$$

So, by 5.4,

$$\lambda_n((X+A)\cap B(0,\tfrac{1}{2}))\leqslant\sum_{i=1}^{2^{2n}k}\lambda_n((X+A_i)\cap B(0,\tfrac{1}{2}))$$

$$<2^{2n}k\frac{\varepsilon}{2^{2n}k}=\varepsilon.$$

Now we are in a position to complete the proof of Theorem 5.1.

(a) ($\supseteq$) Let $A\in\mathfrak{J}^n$ and let k be such that $A\in\mathfrak{J}^n_k$. We claim that there is $\varepsilon>0$ such that, for any $X\subseteq\boldsymbol{R}^n$, if $\lambda_n(X)<\varepsilon$, then $X+A\neq\boldsymbol{R}^n$. By 5.3 there are

$$A_i=\{a^i_j\colon j<\omega\}\in\mathscr{A}\quad\text{for } 1\leqslant i\leqslant 2^{2n}k$$

such that

$$A=\bigcup_{i=1}^{2^{2n}k}A_i.$$

Let

$$\varepsilon<\frac{\lambda_n(B(0,\tfrac{1}{2}))}{2^{2n}k},\quad X\subseteq\boldsymbol{R}^n\quad\text{and}\quad\lambda_n(X)<\varepsilon.$$

Then

$$\lambda_n((X+A_i)\cap B(0,\tfrac{1}{2}))\leqslant\sum_{j\in\omega}\lambda_n(X\cap B(-a^i_j,\tfrac{1}{2}))$$

$$\leqslant\lambda_n(X)<\varepsilon<\frac{\lambda_n(B(0,\tfrac{1}{2}))}{2^{2n}k}.$$

Hence

$$\lambda_n((X+A)\cap B(0,\tfrac{1}{2}))\leqslant\sum_{i=1}^{2^{2n}k}\lambda_n((X+A_i)\cap B(0,\tfrac{1}{2}))$$

$$<2^{2n}k\frac{\lambda_n(B(0,\tfrac{1}{2}))}{2^{2n}k}=\lambda_n(B(0,\tfrac{1}{2})).$$

Thus $X+A\neq\boldsymbol{R}^n$.

(b) Let $A\in\mathfrak{J}^n\cap\mathrm{Bd}^*$ and k be such that $A\in\mathfrak{J}^n_k$. Let $X\in\mathfrak{L}^{<\infty}_n$. Let $a\in\boldsymbol{R}^n$ be such that

$$B(0,t(X,\lambda_n(B(0,\tfrac{1}{2})),k))\cap(A+a)=\varnothing.$$

By 5.5 applied to $A+a$ and $\varepsilon = \lambda_n(B(0, \frac{1}{2}))$ we have
$$\lambda_n((X+A+a) \cap B(0, \tfrac{1}{2})) < \lambda_n(B(0, \tfrac{1}{2})).$$
Hence $X+A \neq \boldsymbol{R}^n$.

COROLLARY 5.6. $g(\mathfrak{L}_n^{<\infty}) = [\boldsymbol{R}^n]^{<\omega}$.

Proof. If $A \in g(\mathfrak{L}_n^{<\infty}) \subseteq (L_n^{<\infty})^*$, then by 5.1 (b) we have $A \in \mathfrak{I}^n$. Therefore, if A is infinite, then A is unbounded. But then $\lambda_n(A+I^n) = \infty$, which contradicts $A \in g(\mathfrak{L}_n^{<\infty})$.

Note that we have
$$g([\boldsymbol{R}]^{<\omega}) = [\boldsymbol{R}]^{<\omega} \subsetneq [\boldsymbol{R}]^{\leqslant\omega} = g([\boldsymbol{R}]^{\leqslant\omega}), \qquad \mathrm{Bd} \subseteq \mathfrak{L}^{<\infty}$$
and
$$g(\mathrm{Bd}) = \mathrm{Bd} \supsetneq [\boldsymbol{R}]^{<\omega} = g(\mathfrak{L}^{<\infty}),$$
so g is not an isotonic operation.

Some open questions:

(1) Does the equality $\mathscr{F}_t([\boldsymbol{R}]^{\leqslant\omega}, \mathfrak{L}) = \mathfrak{L}^*$ hold? **(P 1367)** (F. Galvin asked whether $\mathcal{L}_{\omega_1}(\mathfrak{L}) \subseteq \mathfrak{L}^*$.)

(2) Is $[\boldsymbol{R}]^{\leqslant\omega}$ $**$-closed? **(P 1368)**

(3) Is it consistent that $\mathfrak{K}$ (resp. $\mathfrak{L}$) is $**$-closed? **(P 1369)**

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW, POLAND

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