COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

QUASICONFORMAL EXTENSIONS OF SOME SPECIAL UNIVALENT FUNCTIONS

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1. Introduction. Statement of results. Functions univalent in the unit disk Δ which admit a quasiconformal (abbreviated: qc) extension to the extended plane $\hat{C}$ play an important role in the theory of Teichmüller spaces; cf. [4], [9]. Mappings of this kind, i.e. qc in $\hat{C}$ and conformal in a subdomain of $\hat{C}$, appear already as a tool in the seminal paper of Bojarski [5], but the study of such mappings for their own sake was initiated by Ahlfors and Weill [3] some 20 years ago in connection with Nehari's criterion of univalence.

One of the simplest and most natural problems is to find conditions for a given function f univalent in Δ to have a qc extension to $\hat{C}$ and possibly to give an explicit construction of such an extension. To this end we need the notions of quasicircle and qc reflection due to Ahlfors [2]. Let Γ be a Jordan curve in $\hat{C}$. We call Γ a *quasicircle* if it is the image line of a circle under a qc automorphism of $\hat{C}$. For other equivalent definitions and properties of quasicircles cf. [7]. Let G, G^* be complementary domains of a Jordan curve Γ in $\hat{C}$. We say that φ is a *qc reflection* in Γ if φ is a sense-reversing qc mapping of G onto G^* whose homeomorphic extension to the closure $\bar{G}$ (which necessarily exists) keeps the points on Γ fixed. Obviously, φ may be extended to the whole plane as a sense-reversing qc automorphism of $\hat{C}$ when defined as φ^{-1} in G^*. Thus points $w \in G$, $\varphi(w) = w^* \in G^*$ are called *quasisymmetric* w.r.t. Γ.

A Jordan curve Γ admits a qc reflection φ if and only if it is a quasicircle. For the proof of this statement cf. Lehto–Virtanen book [10] which is our standard reference, as far as qc mappings are concerned. A Jordan domain whose boundary is a quasi-circle is called a *quasidisk*.

We can now state the complete solution of the problem mentioned above. A necessary and sufficient condition for a function f univalent in Δ (or in $\Delta^* = \hat{C} \setminus \bar{\Delta}$) to have a qc extension to the whole plane is the following: f has a homeomorphic extension to the closure $\bar{\Delta}$ (or $\bar{\Delta}^*$, resp.) and the image curve $f(\partial\Delta)$ is a quasicircle. The necessity is trivial. The sufficiency is almost trivial and is quoted as Lemma 1 in the next section. With another simple lemma it yields explicit construction of qc extensions for some special univalent functions. We were able to find explicit qc extensions of functions convex in one direction, spirallike functions, and also to generalize an earlier result [8] on qc extension of close-to-convex functions. For more information about these special univalent functions see [11].

2. Auxiliary results. In this section we shall prove two simple lemmas which will serve as a tool in obtaining explicit qc extensions in Section 3.

LEMMA 1. *Suppose that f maps conformally the unit disk Δ onto a quasidisk G with boundary Γ. If φ is a qc reflection in Γ and σ: $z \mapsto 1/\bar{z}$ is the reflection in $T = \partial\Delta$ then*

$$F = \begin{cases} f & \text{in } \bar{\Delta}, \\ \varphi \circ f \circ \sigma & \text{in } \hat{C} \setminus \Delta \end{cases} \tag{1}$$

is a qc extension of f to $\hat{C}$.

Lemma 1 follows immediately from the fact that F is a homeomorphism of $\hat{C}$ onto itself which is conformal in Δ and qc in Δ^*, the common boundary T of both domains being a removable set; cf. [10]. Obviously an analogous lemma holds for conformal mappings of Δ^*.

LEMMA 2. *Suppose the function p: $\boldsymbol{R} \to \boldsymbol{R}$ satisfies the Lipschitz condition on the real axis $\boldsymbol{R}$:*

$$|p(u_1) - p(u_2)| \leqslant M|u_1 - u_2|, \qquad u_1, u_2 \in \boldsymbol{R}. \tag{2}$$

Then the curve Γ:

$$\{u + iv\colon\ u \in \boldsymbol{R} \wedge v = p(u)\} \tag{3}$$

is a quasicircle and the mapping

$$\varphi\colon\ u + iv \mapsto u + i[2p(u) - v] \tag{4}$$

is a qc reflection in Γ.

Proof. Obviously φ may be written in the form

$$\varphi(w) = \bar{w} + 2ip\left(\tfrac{1}{2}(w + \bar{w})\right), \qquad w = u + iv. \tag{5}$$

It is easily verified that φ is a homeomorphism carrying $G = \{u + iv\colon u \in \boldsymbol{R} \wedge v < p(u)\}$ onto its complementary domain G^* and admitting a homeomorphic extension on Γ that keeps the points on Γ fixed. Hence it is sufficient to show that

$$w \mapsto \bar{\varphi}(w) = \psi(w) = w - 2ip\left(\tfrac{1}{2}(w + \bar{w})\right)$$

is qc in G. For almost all $u \in \boldsymbol{R}$ the derivative p' exists and satisfies $|p'(u)| \leqslant M$. Hence for a corresponding w both formal derivatives of ψ exist and satisfy

$$\psi_{\bar{w}} = -ip'(u), \qquad \psi_w = 1-ip'(u), \qquad u = \operatorname{Re} w.$$

The complex dilatation $\mu_\psi = \psi_{\bar{w}}/\psi_w$ satisfies a.e. in $\hat{C}$

$$|\mu_\psi| = |p'(u)|\{1+p'(u)^2\}^{-1/2} \leqslant M(1+M^2)^{-1/2} < 1$$

and this ends the proof.

3. Applications

3.1. We call a domain G *strongly convex* in the direction $e^{i\beta}$ iff

(i) any straight line parallel to the vector $e^{i\beta}$ intersects both G and its complementary set $C\setminus G$;

(ii) there exists $\delta\in(0, \pi/2)$ such that for any boundary point z_0 of G the angle with vertex at z_0 of measure 2δ, which is bisected by the ray emanating from z_0 and parallel to the vector $e^{i\beta}$, is contained in $C\setminus G$.

Without loss of generality we may assume that $\beta = \pi/2$.

Then the boundary of G is the graph of a function $y = p(x)$, $x\in \mathbf{R}$. Take any x_1, x_2 with $x_1 < x_2$. From (ii) it follows by taking $x_1 = \operatorname{Re} z_0$ that $[p(x_2) - p(x_1)]/[x_2-x_1]\cot\delta$ i.e. $|p(x_2)-p(x_1)| \leqslant |x_2-x_1|\cot\delta$ for $p(x_2) \geqslant p(x_1)$. If $p(x_2) < p(x_1)$ take $x_2 = \operatorname{Re} z_0$ and now (ii) implies $[p(x_1)-p(x_2)]/[x_2 - x_1] \leqslant \cot\delta$ and again $|p(x_2)-p(x_1)| \leqslant |x_2-x_1|\cot\delta$. Hence p is lipschitzian and Lemma 2 provides a qc reflection in ∂G.

A function f is called *strongly convex* in the direction $e^{i\beta}$ iff it maps Δ conformally onto a domain strongly convex in this direction. A suitable rotation and then a subsequent application of Lemmas 1, 2 yield an explicit construction of a qc extension of f to the whole plane $\hat{C}$.

3.2. We call a domain G *spirallike* of type α, $\alpha\in(-\pi/2, \pi/2)$, iff with any $w_0\in G$ the whole arc of the logarithmic spiral $(-\infty, 0]\ni\tau \to w_0\exp(e^{i\alpha}\tau)$ together with its end-points 0, w_0 is contained in G. This implies that with any $w_1\in C\setminus G$ the whole arc $[0, +\infty)\ni\tau\mapsto w_1\exp(e^{i\alpha}\tau)$ is contained in $C\setminus G$.

In analogy with 3.1 a domain G is called *strongly spirallike of type* α_0 iff

(iii) $0\in G$ and any logarithmic spiral $\mathbf{R}\ni\tau\mapsto w_0\exp(e^{i\alpha_0}\tau)$ intersects both G and $C\setminus G$;

(iv) there exists $\delta > 0$ such that G is spirallike of type α for any $\alpha\in[\alpha_0-\delta, \alpha_0+\delta]$.

Evidently, the mapping $w\mapsto\log w$ carries G into a domain H which is strongly convex in the direction $e^{i\alpha_0}$ and invariant under the mapping $W\mapsto W+2\pi i$. The points in H congruent modulo $2\pi i$ correspond to one point in G. Thus after a suitable rotation a qc reflection in ∂H can be constructed as in Lemma 2. The mapping $W\mapsto\exp W$ carrying back congruent pairs of points quasisymmetric w.r.t. ∂H into a pair of points $w\in G$, $w^*\in C\setminus G$ obviously induces a qc reflection φ: $w\mapsto w^*$ in ∂G.

Now, the mapping function $f\colon \Delta \to G$ has a qc extension to the whole plane described by Lemma 1. Note that the function $f(z) = z + a_2 z^2 + \ldots$ is mapping Δ conformally onto a domain G spirallike of type α iff $\operatorname{Re}\{e^{-i\alpha} zf'(z)/f(z)\} > 0$ and $\alpha \in (-\pi/2, \pi/2)$; cf. [11].

3.3. A function f holomorphic in Δ is said to be *close-to-convex* iff there exists a conformal mapping g of Δ onto a convex domain G such that

$$f'(z)/g'(z) = p(z), \quad \operatorname{Re} p(z) > 0 \quad \text{in } \Delta. \tag{6}$$

This condition implies univalence of f in Δ. In fact, if we put

$$h(w) = f \circ g^{-1}(w), \quad w \in G \tag{7}$$

then obviously

$$h'(w) = p \circ g^{-1}(w) = q(w) \tag{8}$$

is a function of positive real part in the convex domain G and this readily implies univalence of h. In [8] an explicit qc extension h^* of h to $\hat{C}$ has been given in case G is a bounded convex domain and the values of q are situated in a compact subset of the right half-plane. In this case g admits, by means of a starlike reflection (cf. [1], [6]), a qc extension g^* to the whole plane so that f admits a qc extension $f^* = h^* \circ g^*$. However, an analogous extension is possible also if G is an unbounded convex domain which is a quasidisk. Then G is either a half-plane, or a convex domain whose boundary has two different asymptotic directions subtending an angle γ, $0 < \gamma < \pi$. In either case G is strongly convex in some direction and hence g admits a qc extension of the form $h^* \circ g^*$ because the construction of h^* for both bounded and unbounded convex quasidisks is the same and was presented in [8]. Therefore we have

THEOREM. *Let f be a close-to-convex function in Δ such that the univalent function g in (6) maps Δ onto a convex quasidisk G and the values of p in (6) are situated in a compact subset of the right half-plane. Given $w \in C \setminus G$ let $\tau = \tau(w)$ be the unique point on ∂G where $|w - \tau|$ attains its minimum as τ ranges over G. Let h^* be defined as follows:*

$$h^*(w) = \begin{cases} h(w), & w \in G, \\ h(\tau(w)) + w - \tau(w), & w \in C \setminus G, \\ \infty, & w = \infty. \end{cases}$$

If g^* is a qc extension of g to $\hat{C}$ then $f^* = h^* \circ g^*$ represents a qc extension of f to $\hat{C}$.

The theorem fails to hold if the convex domain G is not a quasidisk. E.g. for $f(z) = g(z) = \frac{1}{2}\log(1+z)/(1-z)$, $p(z) \equiv 1$, G is the strip $\{w\colon |\operatorname{Im} w| < \pi/4\}$ and obviously f does not admit even a homeomorphic extension to $\hat{C}$.

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Reçu par la Rédaction le 10. 05. 1984