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ON THE GENERAL MINIMAX THEOREM

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0. This paper is devoted to the following problem, the interest of which is well known in the theory of games and some other fields:

Let C and D be two arbitrary sets, and $F(x, y)$ a real function (1) on $C \times D$. Under which conditions we can assert that

$$\inf_{x \in C} \sup_{y \in D} F(x, y) = \sup_{y \in D} \inf_{x \in C} F(x, y)\, ? \tag{0.1}$$

The results we are going to establish will include, as special cases, almost all of the variants of the minimax theorem known up to the present. More specifically, Theorem 1 contains the results of Nikaido [6] and of Wu Wen-tzun [9]; Theorem 2 contains the results of Sion [7] and of Wu Wen-tzun [9] and permits a straightforward derivation of a result of Ky Fan [5]; Theorem 3 provides a generalization of a more recent result of Golshtein-Movshovich [2].

We would like to draw the attention of the reader to the fact that all proofs in this paper are elementary and, unlike most of the common proofs for minimax propositions, make no appeal to theorems like the separation theorem for convex sets (i.e., the Hahn-Banach theorem), the Brouwer fixed-point principle and its generalizations to set-valued mappings, the Helly theorem on intersection of convex sets, and the Sperner lemma and its consequences.

1. Assume that C and D are subsets of two Hausdorff topological spaces X and Y, respectively. Let a be a real number and let

$$D(x) = D_a(x) = \{y \in D \colon F(x, y) \geqslant a\} \quad \text{for every } x \in C.$$

We say that the function $F(x, y)$ is *a-connected* on $C \times D$ if

(i) the set $\bigcap_{i=1}^{k} D(a^i)$ is connected for any finite system $a^1, \ldots, a^k \in C$;

(1) All the results that follow still hold if $F(x, y)$ is allowed to take on the value $-\infty$.

(ii) for any pair $a, b \in C$, there exists a continuous mapping u: $[0, 1] \to C$ such that $u(0) = a$, $u(1) = b$, and, for any λ, μ, μ',

$$(1.1) \qquad 0 \leqslant \mu \leqslant \lambda \leqslant \mu' \leqslant 1 \Rightarrow D\big(u(\lambda)\big) \subset D\big(u(\mu)\big) \cup D\big(u(\mu')\big).$$

Clearly, if X and Y are linear spaces, if C and D are convex subsets of X and Y, respectively, and if the sets $\{y \in D: F(x, y) \geqslant a\}$ and $\{x \in C: F(x, y) < a\}$ for every $x \in C$, $y \in D$ are convex, then $F(x, y)$ is a-connected.

THEOREM 1. *Assume that*

(1) *the set D is compact*;

(2) *the function $F(x, y)$ is a-connected on $C \times D$ for*

$$a = \inf_{x \in C} \sup_{y \in D} F(x, y);$$

(3) *the function $F(x, y)$ is continuous separately in x and y.*

Then equality (0.1) *holds.*

We shall first prove the following

LEMMA 1.1. *Under assumptions of Theorem* 1 *every two sets $D(a)$ and $D(b)$ with $a, b \in C$ have a common element.*

Proof. Since the set D is compact and the function $F(x, y)$ is continuous in y, we have

$$a = \inf_{x \in C} \max_{y \in D} F(x, y),$$

and so the set $D(x)$ for every $x \in C$ is non-empty and closed; moreover, the a-connectedness of $F(x, y)$ implies that $D(x)$ is connected.

Suppose now that the sets $D(a)$ and $D(b)$ for some $a, b \in C$ are disjoint. Let u: $[0, 1] \to C$ be the continuous mapping that corresponds to the pair a, b according to assumption (2). Then the set $D\big(u(\lambda)\big)$ for every $\lambda \in [0, 1]$ cannot meet simultaneously $D(a)$ and $D(b)$; for otherwise, by (1.1), we would have $D\big(u(\lambda)\big) = E_a \cup E_b$, where $E_a = D\big(u(\lambda)\big) \cap D(a)$ and $E_b = D\big(u(\lambda)\big) \cap D(b)$ are two closed, non-empty, disjoint sets, contrary to the connectedness of $D\big(u(\lambda)\big)$.

Thus, for every $\lambda \in [0, 1]$, one and only one of the following alternatives holds:

(a) $D\big(u(\lambda)\big) \subset D(a)$ or (b) $D\big(u(\lambda)\big) \subset D(b)$.

Let M_a and M_b denote the sets of those $\lambda \in [0, 1]$ that satisfy (a) and (b), respectively. Obviously, $0 \in M_a$, $1 \in M_b$ and $M_a \cup M_b = [0, 1]$. On the other hand, in view of assumption (3), it is easy to see that the sets M_a and M_b are open in $[0, 1]$. Indeed, consider an arbitrary $\bar{\lambda} \in [0, 1]$ and suppose that $\bar{\lambda} \in M_a$. It follows that $D\big(u(\bar{\lambda})\big) \subset D(a)$, whence $D\big(u(\bar{\lambda})\big) \cap D(b) = \emptyset$, i.e.

$$\big(\forall y \in D(b)\big) \; F\big(u(\bar{\lambda}), y\big) < a.$$

Using the continuity of $F(x, y)$ in x, we can find, for every fixed $y \in D(b)$, a neighbourhood V_y of $u(\bar{\lambda})$ such that

$$(\forall x \in V_y)\ F(x, y) < a.$$

Since $u^{-1}(V_y)$ is a neighbourhood of $\bar{\lambda}$, there exist numbers $\lambda_i = \lambda_i(y) \in u^{-1}(V_y)$ $(i = 1, 2)$ such that $I_y = [\lambda_1, \lambda_2]$ is still a neighbourhood of $\bar{\lambda}$. We have $F(u(\lambda_i), y) < a$ $(i = 1, 2)$, and hence, by continuity of $F(x, y)$ in y, we can find, for every $i = 1, 2$, a neighbourhood $W_i(y)$ of y satisfying

$$(\forall y' \in W_i(y))\ F(u(\lambda_i), y') < a.$$

Then $W_y = W_1(y) \cap W_2(y)$ is a neighbourhood of y such that, for $i = 1, 2$,

$$(\forall y' \in W_y)\ F(u(\lambda_i), y') < a, \qquad \text{i.e. } y' \notin D(u(\lambda_i)).$$

Therefore, according to (1.1), $y' \notin D(u(\lambda))$ for every $\lambda \in I_y$. We have thus shown that to every $y \in D(b)$ there correspond a neighbourhood W_y of y and a neighbourhood I_y of $\bar{\lambda}$ such that

$$(\forall \lambda \in I_y)\ (\forall y' \in W_y)\ F(u(\lambda), y') < a.$$

Since $D(b)$ is a closed subset of the compact set D, it is itself compact, and so there exists a finite subset Q of $D(b)$ such that the family W_y for $y \in Q$ covers $D(b)$. If

$$\lambda \in I = \bigcap\{I_y\colon\ y \in Q\} \qquad \text{and} \qquad y \in D(b),$$

then $y \in W_{y'}$, for some $y' \in Q$, and hence $F(u(\lambda), y) < a$. Therefore, $D(u(\lambda)) \subset D(a)$ for every $\lambda \in I$, and we have $I \subset M_a$, which means that M_a is open in $[0, 1]$. In a similar way M_b is open in $[0, 1]$.

Thus the segment $[0, 1]$ is the union of two non-empty, disjoint subsets M_a and M_b which are both open in it. Since this is impossible, we must have $D(a) \cap D(b) \neq \emptyset$.

Proof of Theorem 1. The inequality

$$\inf_{x \in C} \sup_{y \in D} F(x, y) \geqslant \sup_{y \in D} \inf_{x \in C} F(x, y)$$

being trivial, we must only prove the converse inequality, i.e.

$$\bigcap_{x \in C} D(x) \neq \emptyset.$$

Since, moreover, D is compact and every $D(x)$ is closed, it suffices to show that the sets $D(x^1), \ldots, D(x^k)$ have a non-empty intersection for every finite system $x^1, \ldots, x^k \in C$. To do this let us proceed by induction.

For $k = 2$ this follows from Lemma 1.1; assume that this holds for $k = h-1$, and prove it for $k = h$. Let $D' = D(x^h)$, $D'(x) = D' \cap D(x)$.

It follows from Lemma 1.1 that the set $D'(x)$ is non-empty for every $x \in C$. That is,

$$\left(\forall x \in C\right) \left(\exists y \in D'\right) \; F(x, y) \geqslant a,$$

and hence

$$a = \inf_{x \in C} \sup_{y \in D'} F(x, y),$$

so that assumptions (1)-(3) of Theorem 1 still hold if we replace D by D'. Therefore, by the inductive assumption, the sets $D'(x^i)$ for $i = 1, \ldots, h-1$ have a non-empty intersection. In other words,

$$\bigcap_{i=1}^{h} D(x^i) \neq \emptyset$$

which was to be shown.

COROLLARY 1.1. *Assume that C and D are convex subsets of two linear topological spaces X and Y, respectively, and that*

(1) *D is compact;*

(2) *for every $x \in C, y \in D$, the sets*

$$D(x) = \{y \in D\colon F(x, y) \geqslant a\} \quad \text{and} \quad C(y) = \{x \in C\colon F(x, y) < a\},$$

where

$$a = \inf_{x \in C} \sup_{y \in D} F(x, y),$$

are convex;

(3) *the function $F(x, y)$ is continuous separately in x and y.*

Then equality (0.1) *holds.*

COROLLARY 1.1′. *Assume that C and D are convex subsets of two linear topological spaces X and Y, respectively, and that*

(1) *C is compact;*

(2) *for every $x \in C, y \in D$, the sets*

$$D^*(x) = \{y \in D\colon F(x, y) > \beta\} \quad \text{and} \quad C^*(y) = \{x \in C\colon F(x, y) \leqslant \beta\},$$

where

$$\beta = \sup_{y \in D} \inf_{x \in C} F(x, y),$$

are convex;

(3) *the function $F(x, y)$ is continuous separately in x and y.*

Then equality (0.1) *holds.*

To prove this proposition it suffices to apply Corollary 1.1 to the function $-F(x, y)$.

COROLLARY 1.2 (Nikaido [6]). *Assume that C and D are convex subsets of two linear topological spaces X and Y, respectively, and that*

(1) *either C or D is compact;*

(2) *$F(x, y)$ is quasiconvex in x and quasiconcave in y;*

(3) *$F(x, y)$ is continuous separately in x and y.*

Then equality (0.1) *holds.*

We recall that a function $f(x)$ on a convex set C is said to be *quasiconvex* if for every real γ the set $\{x \in C\colon f(x) \leqslant \gamma\}$ is convex or, equivalently, if for every real γ the set $\{x \in C\colon f(x) < \gamma\}$ is convex. A function $g(y)$ on a convex set D is said to be *quasiconcave* if its negative is quasiconvex.

Corollary 1.2 is proved by combining Corollaries 1.1 and 1.1′. It should be noticed that this result has been established by Nikaido on the basis of the Brouwer fixed-point principle (a proof based on the Kakutani fixed-point principle can be found in [2]).

The notion of a-connectedness we have defined in Section 1 is closely related to the notion of strong connectedness due to Wu Wen-tzun [9]. Let us, namely, set $D_\eta^*(x) = \{y \in D\colon F(x, y) > \eta\}$ for every real number η. A function $F(x, y)$ is said to be *strongly connected* on the set $C \times D$ if

(W_1) for any finite system $a^1, \ldots, a^k \in C$ and for any real η, the set $\bigcap\limits_{i=1}^{k} D_\eta^*(a^i)$ is connected (possibly empty);

(W_2) for any pair $a, b \in C$, there exists a continuous mapping u: $[0, 1] \to C$ such that $u(0) = a$, $u(1) = b$ and, for any real η and any λ, μ, μ',

$$0 \leqslant \mu \leqslant \lambda \leqslant \mu' \leqslant 1 \Rightarrow D_\eta^*\big(u(\lambda)\big) \subset D_\eta^*\big(u(\mu)\big) \cup D_\eta^*\big(u(\mu')\big). \tag{1.2}$$

The relationship between this notion and that of a-connectedness is summarized in the following

LEMMA 1.2. *If $F(x, y)$ is strongly connected, if D is compact and if $F(x, y)$ is upper semi-continuous in y, then $F(x, y)$ is a-connected for every real a.*

Proof. First we note that, x being an arbitrary element of C, we have

$$D = \bigcup\{D_\eta^*(x)\colon \eta < \gamma\}, \qquad \text{where } \gamma = \max\{F(x, y)\colon y \in D\}$$

(the maximum is attained because of the compactness of D and the upper semi-continuity of $F(x, y)$). Since every $D_\eta^*(x)$ is connected by (W_1) and since the intersection of the family $\{D_\eta^*(x)\colon \eta < \gamma\}$ contains the non-empty set $\{y \in D\colon F(x, y) = \gamma\}$, it follows, by a well-known property of connected sets, that D is itself connected. Consider now any finite system $a^1, \ldots, a^k \in C$. Since every function $F(a^i, y)$ is upper semi-continuous, so is $f(y) = \min\limits_{1 \leqslant i \leqslant k} F(a^i, y)$. If the set

$$E = \bigcap_{i=1}^{k} D_a(a^i)$$

is not connected, then $E = E_1 \cup E_2$ with E_1, E_2 closed, disjoint and non-empty. Since every compact space is normal, there exist two disjoint open sets G_1, G_2 such that $G_i \supset E_i$ $(i = 1, 2)$, and since the set $D \setminus (G_1 \cup G_2)$ is non-empty (because of the connectedness of D) and compact (as a closed subset of the compact set D), the function $f(y)$ attains a maximum η on $D \setminus (G_1 \cup G_2)$. By the definition of E, we have $f(y) < a$ for every $y \notin E$, so that $\eta < a$, and hence

$$E_i \subset E^* = \bigcap_{i=1}^{k} D^*(a^i) \qquad (i = 1, 2).$$

Therefore, $E^* \cap G_i \supset E_i \neq \emptyset$ $(i = 1, 2)$, and since $E^* \subset G_1 \cup G_2$, this means that E^* is not connected, contrary to condition (W_1). Thus, under the assumption stated in Lemma 1.2, condition (i) in the definition of a-connectedness is satisfied. On the other hand, if $a, b \in C$ and if u is the mapping associated with them by (W_2), then we must have (1.1). Indeed, if $F(u(\mu), y) < a$ and $F(u(\mu'), y) < a$ for some $y \in D$, then there exists $\eta < a$ such that $F(u(\mu), y) \leqslant \eta$ and $F(u(\mu'), y) \leqslant \eta$ and, therefore, by (1.2) we can write $F(u(\lambda), y) \leqslant \eta < a$ for any $\lambda \in [\mu, \mu']$. In other words, if $y \notin D(u(\mu)) \cup D(u(\mu'))$, then $y \notin D(u(\lambda))$, as required by (1.1).

From Lemma 1.2 and Theorem 1 we deduce

COROLLARY 1.3 (Wu Wen-tzun). *Assume that*

(1) *the set D is compact*;

(2) *the function $F(x, y)$ is strongly connected*;

(3) *the function $F(x, y)$ is continuous separately in x and y.*

Then equality (0.1) *holds.*

Actually, in [9] Wu Wen-tzun assumed also that the space Y is separable (and made use of this assumption in his proof which relies on the 1-dimensional form of the Helly theorem). As follows from the foregoing, this separability assumption may be dropped. Moreover, as will be seen shortly, assumption (3) in Corollaries 1.2 and 1.3 can be replaced by a weaker one.

2. We now try to relax the continuity condition (3) in Theorem 1. The price we must pay, however, is some strengthening of the connectedness condition (2).

We say that a function $F(x, y)$ is *a-strongly connected* on $C \times D$ if there exists a sequence $\varepsilon_s \downarrow 0$ $(s = 1, 2, \ldots)$ such that, for every s, the function $F(x, y) + \varepsilon_s$ is a-connected on $C \times D$.

THEOREM 2. *Assume that*

(1) *the set D is compact*;

(2) *the function $F(x, y)$ is a-strongly connected on $C \times D$ for*

$$a = \inf_{x \in C} \sup_{y \in D} F(x, y);$$

(3) *the function* $F(x, y)$ *is lower semi-continuous in* x *and upper semi--continuous in* y.

Then equality (0.1) *holds.*

For every $x \in C$ and every s we set

$$D^s(x) = D_{a-\varepsilon_s}(x) = \{y \in D\colon F(x, y) \geqslant a - \varepsilon_s\}.$$

LEMMA 2.1. *Under assumptions* (1) *and* (3) *of Theorem* 2, *if* $F(x, y)$ *is* $(a-\varepsilon_s)$*-connected on* $C \times D$, *then every two sets* $D^s(a)$ *and* $D^s(b)$ *with* $a, b \in C$ *have a common element.*

Proof. Assume that $D^s(a) \cap D^s(b) = \emptyset$ for some $a, b \in C$. By the same argument as that used in the proof of Lemma 1.1, but replacing $F(x, y)$ by $F(x', y) + \varepsilon_s$, we can see that if $u\colon [0, 1] \to C$ is the mapping that corresponds to the pair a, b for $F(x, y) + \varepsilon_s$, then, for every $\lambda \in [0, 1]$, one and only one of the following alternatives holds:

(a) $D^s(u(\lambda)) \subset D^s(a)$ or (b) $D^s(u(\lambda)) \subset D^s(b)$.

Let M_a and M_b denote the sets of those $\lambda \in [0, 1]$ that satisfy (a) and (b), respectively. Obviously, $0 \in M_a$, $1 \in M_b$ and $M_a \cup M_b = [0, 1]$. Furthermore, if $\mu \in M_a$ and $\mu' \in M_a$, then $[\mu, \mu'] \subset M_a$, since

$$D^s(u(\lambda)) \subset D^s(u(\mu)) \cup D^s(u(\mu')) \quad \text{for every } \lambda \in [\mu, \mu'].$$

Let $\bar{\lambda} = \sup M_a = \inf M_b$ and suppose, for example, that $\bar{\lambda} \in M_a$ (the case $\bar{\lambda} \in M_b$ can be treated similarly). Then $D^s(u(\bar{\lambda})) \subset D^s(a)$, i.e.

$$(2.1) \qquad (\forall y \notin D^s(a))\ F(u(\bar{\lambda}), y) + \varepsilon_s < a.$$

If $\{\lambda_m\}$ denotes a sequence in M_b which converges to $\bar{\lambda}$, then, for every m, we have $D^s(u(\lambda_m)) \subset D^s(b)$, so that

$$(\forall y \in D^s(a))\ F(u(\lambda_m), y) + \varepsilon_s < a.$$

By letting $m \to \infty$ and using the lower semi-continuity of $F(x, y)$ in x, we then conclude

$$(2.2) \qquad (\forall y \in D^s(a))\ F(u(\bar{\lambda}), y) + \varepsilon_s \leqslant a,$$

which together with (2.1) implies

$$\sup_{y \in D} F(u(\bar{\lambda}), y) \leqslant a - \varepsilon_s < a,$$

contrary to the definition of a. Thus the hypothesis $D^s(a) \cap D^s(b) = \emptyset$ is untenable and the proof of Lemma 2.1 is complete.

Proof of Theorem 2. We first observe that if $\bar{a} \in C$, then, setting $D' = D^s(\bar{a})$, by Lemma 2.1 we have $D^{s'}(\bar{a}) \cap D^{s'}(x) \neq \emptyset$ for every $x \in C$

and every $s' > s$. Hence, since $D^{s'}(\bar{a}) \subset D^s(\bar{a})$, we have $D^s(\bar{a}) \cap D^{s'}(x) \neq \emptyset$. Therefore,

$$(\forall x \in C) \; (\exists y \in D') \; F(x, y) \geqslant a - \varepsilon_{s'},$$

which means that

$$\inf_{x \in C} \sup_{y \in D'} F(x, y) = a.$$

From this remark and Lemma 2.1 it follows, by an argument analogous to that used in the proof of Theorem 1, that

$$\bigcap_{x \in C} D^s(x) \neq \emptyset \quad \text{for every } s.$$

Let y^s be an element belonging to all $D^s(x)$, $x \in C$, and let y^0 be a cluster point of the sequence $\{y^s\}$. Since

$$(\forall s) \; F(x, y^s) \geqslant a - \varepsilon_s \quad \text{for every fixed } x \in C,$$

it follows from the upper semi-continuity of the function $F(x, y)$ in y that $F(x, y^0) \geqslant a$. Therefore,

$$\inf_{x \in C} F(x, y^0) \geqslant a,$$

and hence

$$\sup_{y \in D} \inf_{x \in C} F(x, y) \geqslant a,$$

which was to be proved.

COROLLARY 2.1. *Assume that C and D are convex subsets of two linear topological spaces X and Y, respectively, and that*

(1) *D is compact;*

(2) *there exists a sequence $\varepsilon_s \downarrow 0$ such that, for every s and every $x \in C$, $y \in D$, the sets*

$$D^s(x) = \{y \in D \colon F(x, y) \geqslant a - \varepsilon_s\} \quad \textit{and} \quad C^s(y) = \{x \in C \colon F(x, y) < a - \varepsilon_s\},$$

where

$$a = \inf_{x \in C} \sup_{y \in D} F(x, y),$$

are convex;

(3) *the function $F(x, y)$ is lower semi-continuous in x and upper semi--continuous in y.*

Then equality (0.1) *holds.*

We might formulate Corollary 2.1′ which would be corresponding to Corollary 2.1 just in the same way as Corollary 1.1′ corresponds to Corollary 1.1. By combining Corollaries 2.1 and 2.1′ we obtain

COROLLARY 2.2 (Sion [7]). *Assume that C and D are convex subsets of two linear topological spaces X and Y, respectively, and that*

(1) *either C or D is compact*;

(2) *$F(x, y)$ is quasiconvex in x and quasiconcave in y*;

(3) *$F(x, y)$ is lower semi-continuous in x and upper semi-continuous in y.*

Then equality (0.1) *holds.*

This proposition has been proved in [7] on the basis of the Kuratowski-Knaster-Mazurkiewicz theorem, and in [1] on the basis of the separation theorem.

From Lemma 1.2 and Theorem 2 it also follows that in Corollary 1.3 (theorem of Wu Wen-tzun) condition (3) on the continuity of $F(x, y)$ can be replaced by a weaker one, namely: $F(x, y)$ is lower semi-continuous in x and upper semi-continuous in y.

Thus Theorem 2 contains, as special cases, both the result of Sion (which, in turn, includes the results of Nikaido, Nash and Kneser) and that of Wu Wen-tzun. Furthermore, it can be used to deduce easily the following result of Ky Fan [5]:

A function $F(x, y)$ defined on $C \times D$ is said to be *pseudoconvex* in x if, for every pair $x^1, x^2 \in C$ and every real $\lambda \in [0, 1]$, there exists an $x \in C$ such that

$$(\forall y \in D)\ F(x, y) \leqslant \lambda F(x^1, y) + (1-\lambda) F(x^2, y).$$

This function is said to be *pseudoconcave* in y if $-F(x, y)$ is pseudoconvex in y.

COROLLARY 2.3 (Ky Fan [5]). *Assume that*

(1) *the set D is compact*;

(2) *the function $F(x, y)$ is pseudoconvex in x and pseudoconcave in y*;

(3) *the function $F(x, y)$ is upper semi-continuous in y.*

Then equality (0.1) *holds.*

Proof. First we note the following property of pseudoconvex-concave functions which is an easy consequence of the minimax theorem:

(*) *If, for some $x^1, \ldots, x^k \in C$, the system $F(x^i, y) \geqslant 0$ $(i = 1, \ldots, k)$ has no solution in D, then there exists an $\bar{x} \in C$ such that $F(\bar{x}, y) \leqslant 0$ for all $y \in D$.*

Indeed, let us consider the set

$$E = \{z = (z_1, \ldots, z_k) \in R^k\colon (\exists y \in D)\ F(x^i, y) + z_i \geqslant 0,\ i = 1, \ldots, k\}.$$

Since no $y \in D$ exists such that $F(x^i, y) \geqslant 0$ $(i = 1, \ldots, k)$, we have $0 \notin E$, i.e. $(\forall z \in E)\ (\exists i)\ z_i > 0$. Therefore, if S^k denotes the simplex

$$\Big\{t \in R^k\colon t_i \geqslant 0, \sum_{i=1}^{k} t_i = 1\Big\},$$

then $(\forall z \in E)\ (\exists t \in S^k)\ \langle t, z\rangle > 0$, which means that

$$\inf_{z \in E} \sup_{t \in S^k} \langle t, z\rangle \geqslant 0 .$$

But, as can be easily verified, the pseudoconcavity of $F(x, y)$ in y implies the convexity of the set E, so that all conditions of Theorem 2 are satisfied for the function $\langle z, t\rangle$ defined on $E \times S^k$. Consequently,

$$\sup_{t \in S^k} \inf_{z \in E} \langle t, z\rangle \geqslant 0, \qquad \text{i.e.}\ (\exists \bar{t} \in S^k)\ (\forall z \in E)\ \langle \bar{t}, z\rangle \geqslant 0$$

and, taking for every $y \in D$ the vector z with $z_i = -F(x^i, y)$ $(i = 1, \ldots, k)$, we have

$$\sum_{i=1}^{k} \bar{t}_i F(x^i, y) \leqslant 0 \qquad \text{for all } y \in D.$$

Since, on the other hand, $F(x, y)$ is pseudoconvex in x, there exists an $\bar{x} \in C$ such that

$$(\forall y \in D)\ F(\bar{x}, y) \leqslant \sum_{i=1}^{k} \bar{t}_i F(x^i, y) \leqslant 0,$$

proving (*).

Turning to the proof of Corollary 2.3, let us take an arbitrary sequence $\varepsilon_s \downarrow 0$. For every s and every finite system $x^1, \ldots, x^k \in C$, the sets $D^s(x^1), \ldots, D^s(x^k)$ must have a non-empty intersection, since otherwise, in view of (*), there would exist an $\bar{x} \in C$ such that

$$(\forall y \in D)\ F(\bar{x}, y) \leqslant a - \varepsilon_s < a,$$

contrary to the definition of a in Theorem 1. Thus the family $D^s(x)$, $x \in C$, has the finite intersection property. Since D is compact and since every set $D^s(x)$ is closed (because of assumption (3)), this family has a non-empty intersection. That is, for each s there exists an element $y^s \in \bigcap_{x \in C} D^s(x)$. If y^0 denotes a cluster point of the sequence $\{y^s\}$, then

$$(\forall x \in C)\ F(x, y^s) \geqslant a - \varepsilon_s,$$

and hence, by the upper semi-continuity of $F(x, y)$ in y, $F(x, y^0) \geqslant a$, i.e.

$$\sup_{x \in C} \inf_{y \in D} F(x, y) \geqslant a,$$

which completes the proof.

3. We conclude the paper by a minimax theorem for the case where X is a locally compact space. In this case, assumption (1) of Theorems 1 and 2 can be somewhat weakened — which may be of interest, since in some applications the requirement that D (or C) be compact turned out to be too stringent.

Let $f_E(x) = \sup\{F(x, y)\colon y \in E\}$ for each subset E of D.

THEOREM 3. *Assume that*

(1) *the space X is locally compact, the set C is closed and the set*

$$C^* = \{x \in C\colon f_D(x) = \inf_{x' \in C} f_D(x')\}$$

is non-empty and compact;

(2) *for every finite subset Q of D, there exists a compact set E such that $Q \subset E \subset D$ and $F(x, y)$ is a'-connected on $C \times E$ for all $a' \in (a-\varepsilon, a+\varepsilon)$, where $a = \inf_{x \in C} \sup_{y \in D} F(x, y)$ and ε is some positive number independent of Q;*

(3) *the function $F(x, y)$ is lower semi-continuous in x and upper semi--continuous in y.*

Then equality (0.1) *holds.*

For the proof of this theorem we need

LEMMA 3.1. *Let $F(x, y)$ be lower semi-continuous in $x \in C$ and let P be a compact subset of C. If*

$$\sup_{y \in D} F(x, y) > \gamma \quad \text{for all } x \in P,$$

then there exists a finite subset Q of D such that

$$\sup_{y \in Q} F(x, y) > \gamma \quad \text{for all } x \in P.$$

Proof. By hypothesis for every $x \in P$ we have $F(x, y_x) > \gamma$ for some $y_x \in D$. Since $F(x', y_x)$ is lower semi-continuous in x', there exists a neighbourhood $V(x)$ of x such that

$$\big(\forall x' \in V(x)\big)\ F(x', y_x) > \gamma.$$

Since P is compact, there exists a finite set $P_1 \subset P$ such that the family $V(x)$, $x \in P_1$, covers P. Let us take $Q = \{y_x, x \in P_1\}$. Then, for every $x \in P$, since $x \in V(x')$ for some $x' \in P_1$, we have $F(x, y_{x'}) > \gamma$, and hence

$$\sup_{y \in Q} F(x, y) > \gamma.$$

Proof of Theorem 3. Since C is a closed subset of X, we may, without loss of generality, assume that $C = X$. From the definition of a and C^* it follows that

$$\big(\forall x \in C^*\big)\ f_D(x) = a \quad \text{and} \quad \big(\forall x \notin C^*\big)\ f_D(x) > a.$$

Since the space X is locally compact, there is a compact neighbourhood K of C^*. Let γ be an arbitrary number satisfying $a - \varepsilon < \gamma < a$.

We have $(\forall x \in K)\ f_D(x) > \gamma$, and so, by Lemma 3.1, there is a finite subset Q_1 of D such that $(\forall x \in K)\ f_{Q_1}(x) > \gamma$. Let us take an open neighbourhood G of C^* contained in K and a compact neighbourhood K' of K. Then $C' = K' \setminus G$ is also compact and $(\forall x \in C')\ f_D(x) > a$, so that, again by Lemma 3.1, there is a finite subset Q_2 of D such that $(\forall x \in C')\ f_{Q_2}(x) > a$. Let E be a compact subset of D which corresponds to $Q = Q_1 \cup Q_2$ according to assumption (2). Obviously,

$$(5.1)\qquad (\forall x \in K)\ f_E(x) \geqslant f_{Q_1}(x) > \gamma,$$

$$(5.2)\qquad (\forall x \in C')\ f_E(x) \geqslant f_{Q_2}(x) > a,$$

$$(5.3)\qquad (\forall x \in C^*)\ f_E(x) \leqslant f_D(x) = a.$$

On the other hand, it is easy to see that

$$(5.4)\qquad (\forall x \notin K')\ f_E(x) > a.$$

Indeed, let $\bar{x} \notin K'$. Since $f_E(x)$ is lower semi-continuous, it attains a minimum η on the compact set C'. Let x^* be an arbitrary element of C^*, and η' a number such that $a < \eta' < \eta$ (see (5.2)). Let $u\colon [0,1] \to C$ be the mapping which is associated with the pair $\bar{x}, x^*$, according to the η'-connectedness of $F(x, y)$. Since $u^{-1}(G)$ and $u^{-1}(X \setminus K')$ are open, their union cannot be equal to $[0,1]$. Consequently, there exists a $\lambda \in [0,1]$ such that $x' = u(\lambda) \in K' \setminus G = C'$. We have $F(x', y) > \eta'$ for some $y \in E$; therefore, by a property of u (see (1.1)), either $F(x^*, y) \geqslant \eta'$ or $F(\bar{x}, y) \geqslant \eta'$. But, according to (5.3), we obtain $F(x^*, y) \leqslant a < \eta'$, and so we must have $F(\bar{x}, y) \geqslant \eta' > a$, which proves (5.4).

From (5.1), (5.2) and (5.4) we then deduce

$$(\forall x \in C)\ f_E(x) > \gamma, \qquad \text{i.e. } \gamma \leqslant \inf_{x \in C} \sup_{y \in E} F(x, y) \leqslant a.$$

Now, $F(x, y)$ being a'-connected on $C \times E$ for every $a' \in (\gamma, a)$ and E being compact, Theorem 2 applies and yields

$$\inf_{x \in C} \sup_{y \in E} F(x, y) = \sup_{y \in E} \inf_{x \in C} F(x, y).$$

Therefore,

$$\sup_{y \in E} \inf_{x \in C} F(x, y) \geqslant \gamma$$

and, *a fortiori*,

$$\sup_{y \in D} \inf_{x \in C} F(x, y) \geqslant \gamma,$$

which implies

$$\sup_{y \in D} \inf_{x \in C} F(x, y) = a,$$

since γ can be taken arbitrarily close to a. The proof is complete.

COROLLARY 3.1. *Let C and D be convex closed subsets of two linear topological spaces X and Y, respectively. Assume that*

(1) *X is finite-dimensional and*

$$C^* = \{x \in C\colon f_D(x) = \inf_{x' \in C} f_D(x')\}$$

is a non-empty compact set;

(2) *$F(x, y)$ is quasiconvex in x and quasiconcave in y;*

(3) *$F(x, y)$ is lower semi-continuous in x and upper semi-continuous in y.*

Then equality (0.1) *holds.*

This follows from the fact that, for any given finite subset Q of D, the convex hull of Q, obviously, yields the compact set E required in assumption (2) of Theorem 3.

Corollary 3.1 contains, as a special case, a result established in [2]. The previous method of the proof is a generalization and a simplification of a method used in [2] for the case $X = R^m$ and $Y = R^n$.

Addendum. After this paper had been written, we remarked that certain results in the first part could be improved. Some of these improvements have been presented in [10] which is, essentially, a shortened version of the first part of this paper. A major improvement consists in noting that Theorem 1 holds even if the function $F(x, y)$ is only upper semi-continuous in each variable (this can be easily seen from the proof given above). Furthermore, Theorems 1 and 2 of the present paper, as well as results in [10], remain in force if the α-connectedness condition in each of them is replaced by the following weaker one (which does not imply the connectedness of the sets $D_\alpha(x)$ as in part (i) of the definition of the α-connectedness).

Let

$$D_\eta(x^1, \ldots, x^k) = \{y \in D\colon F(x^i, y) \geqslant \eta,\ i = 1, \ldots, k\}$$

(in game-theoretical terminology this is the set of all strategies which guarantee to the second player a pay-off not less than η if the first player chooses one of the strategies $x^1, \ldots, x^k$). Then there exists a non-decreasing sequence

$$\eta_n \to \alpha = \inf_{x \in C} \sup_{y \in D} F(x, y)$$

such that to every n, every pair $a, b \in C$, every finite system $x^1, \ldots, x^k \in C$ satisfying $\overline{D'_{\eta_n}(a)} \cap \overline{D'_{\eta_n}(b)} = \emptyset$ – where $D'(x)$ stands for $D(x^1, \ldots, x^k, x)$ and the bar denotes the topological closure operation – there corresponds a continuous mapping $u\colon [0, 1] \to C$ verifying $u(0) = a$, $u(1) = b$ such that, for every interval $[t_0, t_1] \subset [0, 1]$ and every $t \in [t_0, t_1]$, we have either $D'_{\eta_n}(u(t)) \subset D'_{\eta_n}(u(t_0))$ or $D'_{\eta_n}(u(t)) \subset D'_{\eta_n}(u(t_1))$.

REFERENCES

[1] C. Berge, *Topological spaces*, Edinburgh 1963.
[2] Е. Г. Гольштейн, *Теория двойственности в математическом программировании и ее приложения*, Москва 1971.
[3] S. Karlin, *Mathematical methods and theory in games, programming and economics*, London-Paris 1959.
[4] H. Kneser, *Sur un théorème fondamental de la théorie des jeux*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, 234 (1952), p. 2418-2420.
[5] Ky Fan, *Minimax theorems*, Proceedings of the National Academy of Sciences of the U.S.A. 39 (1953), p. 42-47.
[6] H. Nikaido, *On von Neumann's minimax theorem*, Pacific Journal of Mathematics 4 (1954), p. 65-72.
[7] M. Sion, *On general minimax theorems*, ibidem 8 (1958), p. 171-176.
[8] J. von Neumann, *Zur Theorie der Gesellschaftsspiele*, Mathematische Annalen 100 (1928), p. 295-320.
[9] Wu Wen-tzun, *A remark on the fundamental theorem in the theory of games*, Science Record 3 (1959), p. 229-233.
]10] Х. Туи, *Об одной общей минимаксной теореме*, Доклады Академии наук СССР 219 (1974), p. 818-821.

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