COLLOQUIUM MATHEMATICUM

VOL. XXII 1971 FASC. 2

ON THE FAILURE OF A DECOMPOSITION

BY

D. W. SOLOMON (MILWAUKEE, WISCONSIN)

Let $I = [a, b]$ be a non-degenerate, bounded, closed interval of the real line. It is well known that if F is of bounded variation (BV*) on I, then $f(x) = D_x F$ exists almost everywhere on I and is Lebesgue integrable on I. Further, $F = \int f dx + F_S$, where F_S is singular. A question posed some time back and, to our knowledge as yet unanswered, is whether the corresponding parallel theorem holds for generalized bounded variation functions and the Denjoy integrals. More precisely, suppose F is continuous and BVG* on I (see, e.g., [1]). Then it is well known that $f(x) = D_x F$ exists a.e. and is finite. One asks whether f is Denjoy integrable (either wide-sense or restricted-sense). If the answer were in the affirmative, this would yield a Denjoy-integral decomposition theorem parallel to the Lebesgue decomposition theorem cited above. We shall show that, in general, f need not be Denjoy integrable. A further conjecture might be that if f is Denjoy wide-sense integrable and has a BVG* primitive, then f is Denjoy restricted-sense integrable. We shall show that this second conjecture is also false. The reasons for the failures of these two conjectures are very similar, and are obvious from the examples given.

I. We begin by presenting a (discontinuous at one) BVG* function whose derivative is not Denjoy integrable. Let $I = [0, 1]$, $E = \{1-(\frac{1}{2})^n\}_{n=0}^{\infty} = \{a_n\}_{n=0}^{\infty}$. Set $I_n = [a_{n-1}, a_n)$, $n = 1, 2, \ldots$,

$$F(x) = \sum_{n=1}^{\infty} [2^n(x-a_{n-1})+n-1]\chi_{I_n}(x)$$

(χ_E is the characteristic function of E). One readily verifies that F is BVG* on $[0, 1]$, but is discontinuous at $x = 1$. Let $f(x) = D_x F$ where this has meaning, and set $f(x) = 0$ where $D_x F$ does not exist. Since F is AC* (restricted-sense absolutely continuous) on every closed interval not containing $x = 1$, one may verify that the only interval function which can represent the Denjoy integral of f is that generated by F itself. However, since

$$\lim_{x\to 1-0} F(x) = +\infty,$$

it follows that F cannot be a Denjoy integral (alternatively, since $f \geqslant 0$ almost everywhere, if f is Denjoy integrable, it must be Lebesgue integrable, and clearly $x = 1$ is a Lebesgue singular point).

To produce a continuous BVG* function with almost everywhere the same derivative as F, we shall construct a generalized singular function S such that

$$\lim_{x \to 1} [F(x) - S(x)] = 0$$

and S is continuous everywhere except at $x = 1$ (a function S is *generalized singular* if it is BVG* with zero derivative a.e.). Let n be fixed. Let P_n be a partition of I_n so that on the closed intervals $I_{nm} = [a_{nm}, a_{nm+1}]$ associated with this partition, $O(F, I_{nm}) < (\frac{1}{2})^n$ ($O(F, J)$ is oscillation of F on J). There is a continuous, monotone singular function S_{nm} defined on I_{nm} such that $S_{nm}(a_{nm}) = F(a_{nm})$, $S_{nm}(a_{nm+1}) = F(a_{nm+1})$. On defining

$$S(x) = \sum_{n,m} S_{nm}(x) \chi(x)_{[a_{nm}, a_{nm+1})},$$

we have the required generalized singular function. The function $H(x) = F(x) - S(x)$ is the required continuous BVG* function whose derivative is not Denjoy integrable. (An alternate choice of F could have been, e.g., $F(x) = \tan x$, with $I = [0, \pi/2]$. The procedure to obtain S is precisely the same.)

II. We proceed next to present an example of a function f which is Denjoy wide-sense integrable, but which is not Denjoy restricted-sense integrable, although it has a continuous BVG* primitive. Set $I = [0, 1]$, and let C be the Cantor ternary set. Let $\{I_{nm}\}$ be the finite sequence of intervals contiguous to C obtained at the n-th stage of construction of the Cantor set, $I_{nm} = [a_{nm}, b_{nm}]$. Let c_{nm} be the mid-point of I_{nm}. Set F_{nm} equal to the linear function which connects the point $(a_{nm}, 0)$ to the point $(c_{nm}, 1/n)$ on $[a_{nm}, c_{nm}]$ and equal to the linear function which connects the point $(c_{nm}, 1/n)$ to the point $(b_{nm}, 0)$ on $[c_{nm}, b_{nm}]$. Put

$$F(x) = \sum_{n,m} F_{nm}(x) \chi_{I_{nm}}(x).$$

One readily verifies that F is ACG on I and that $f(x) = D_x F$ exists almost everywhere and is finite. However, F is not ACG* on I. For, each interval J which contains a point of C as interior point contains an interval $I_{nm(n)}$ for all but a finite number of n. Therefore the sum of the oscillations of F over the intervals contiguous to C in J is never finite. Hence F is a "pure" Denjoy wide-sense integral ([1]) and because of the containment

([1]) S. Saks, *Theory of the integral*, 2nd ed., Warszawa 1937, Theorem VII. 8.5, p. 232.

of the restricted-sense integral in the wide-sense integral, f is not restricted-sense integrable.

We shall construct a continuous, wide-sense generalized singular function S such that $F-S$ is BVG*. Let m_n be the number of intervals obtained at the n-th stage of the Cantor construction. On I_{mn} we define S to be a continuous singular function the absolute value of whose difference from F is everywhere on I_{mn} less than $1/(m_n \cdot n^2)$, and which vanishes at a_{nm} and b_{nm}. Let S vanish at all points of C. Set $H = F-S$. Then one easily sees that H is BV on C and

$$\sum_{m,n} O(H, I_{mn}) \leqslant 2 \sum 1/n^2.$$

It follows [2] that H is BV* on C. Since H is clearly BV* on each interval contiguous to C, and since H is easily seen to be continuous, it follows that H is the required BVG* primitive of f.

In closing, we might point out once again for emphasis that F given in part II is an example of a "pure" Denjoy wide-sense integral whose ordinary derivative exists a.e. and is finite [3].

---

(2) Op. cit., Theorem 8.5, p. 232.
(3) Op. cit., Theorem 10.5, p. 235.

Reçu par la Rédaction le 21. 5. 1969