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ON ASYMPTOTIC BEHAVIOUR OF LOCAL MARTINGALES

BY

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1. Let $(\Omega, \mathscr{F}; P)$ be a complete probability space and let $(\mathscr{F}_t)_{t\in\boldsymbol{R}_+}$ be an increasing family of σ-fields of events, satisfying the usual conditions, i.e. $\mathscr{F}_0$ contains all P-negligible sets and

$$\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s \quad \text{for each } t \geqslant 0.$$

In what follows, martingales, local martingales, submartingales, and stopping times will be considered with respect to this family only. All processes considered are assumed to be a.s. right continuous, with finite limits from the left ("cadlag").

The needed definitions can be found, for example, in [5].

Let $M = (M_t)_{t\in\boldsymbol{R}_+}$ be a local martingale ($M_0 = 0$) such that

$$\mathrm{E}\big((\Delta M_T)^2 \chi_{\{T<\infty\}}\big) < \infty \tag{1.1}$$

for every stopping time T, where $\Delta M_t = M_t - M_{t-}$, and χ_A denotes the indicator function of the set A. If $(S_n)_{n=1,2,\ldots}$ is a sequence of stopping times reducing M, $S_n \nearrow \infty$, then the stopping times

$$T_n = S_n \wedge \inf\{t\colon |M_t| \geqslant n\}$$

have the property that $(M_{t\wedge T_n})_{t\in\boldsymbol{R}_+}$ are square-integrable martingales for $n = 1, 2, \ldots$ In other words, (1.1) implies that M is locally square-integrable. Let $\langle M\rangle = (\langle M\rangle_t)_{t\in\boldsymbol{R}_+}$ denote the unique (up to indistinguishability) predictable, increasing process such that

$$M^2 - \langle M\rangle \text{ is a local martingale, } \langle M\rangle_0 = 0. \tag{1.2}$$

Another important increasing process associated with M is defined by

$$[M]_t = \langle M^c\rangle_t + \sum_{s\leqslant t} (\Delta M_s)^2, \tag{1.3}$$

where M^c is the continuous part of M (cf. [5], for the sake of brevity we write $\langle M\rangle$ and $[M]$ instead of $\langle M, M\rangle$ and $[M, M]$, respectively). In the present paper some asymptotic properties of M are established in terms of the processes $\langle M\rangle$ and $[M]$.

PROPOSITION 1. *Under assumption* (1.1)

(1.4) $\{\lim_{t\to+\infty} M_t$ *exists and is finite*$\}$

$$\equiv \{\langle M\rangle_\infty < \infty\} = \{[M]_\infty < \infty\} \text{ a.s.}$$

Moreover,

(1.5) $$\limsup_{t\to+\infty} M_t = +\infty = -\liminf_{t\to+\infty} M_t \text{ a.s.}$$

$$\text{on } \{\langle M\rangle_\infty = +\infty\} \ (\equiv \{[M]_\infty = +\infty\}).$$

We shall need the following simple lemma:

LEMMA 1. *Let M be a local martingale, $M_0 = 0$. If there exists a sequence $(T_n)_{n=1,2,\ldots}$ of stopping times reducing M, $T_n \nearrow \infty$, such that*

(1.6) $$\sup_n \mathrm{E}(M_{T_n}^+) < \infty,$$

then $\lim_{t\to+\infty} M_t$ exists a.s. and is integrable.

Proof. Let $U_a^b(n)$ denote the number of upcrossings of the interval $[a, b]$ by the stopped martingale $(M_{t\wedge T_n})_{t\in\mathbf{R}_+}$ (cf., e.g., [4]). We have the "upcrossing inequality"

$$\mathrm{E}\big(U_a^b(n)\big) \leqslant \frac{1}{b-a}\big(\sup_t \mathrm{E}(M_{t\wedge T_n}^+) + a^-\big) = \frac{1}{b-a}\big(\mathrm{E}(M_{T_n}^+) + a^-\big).$$

Next, if U_a^b denotes the number of upcrossings of $[a, b]$ by $(M_t)_{t\in\mathbf{R}_+}$, then $U_a^b(n) \nearrow U_a^b$, and we get

$$\mathrm{E}(U_a^b) \leqslant \frac{1}{b-a}\big(\sup_n \mathrm{E}(M_{T_n}^+) + a^-\big).$$

The proof is now completed by the standard argument.

Remarks. (a) If (1.6) is satisfied for one sequence $(T_n)_{n=1,2,\ldots}$ of stopping times such that T_n reduces M and $T_n \nearrow \infty$, then it is satisfied for every such sequence.

(b) If instead of (1.6) we assume that

$$\sup_n \mathrm{E}(|M_{T_n}|^p) < \infty \quad \text{for some } p > 1,$$

then, by an immediate extension of Doob's L^p-inequality, M is an (L^p-bounded) martingale.

Proof of Proposition 1. The proof is rather typical (cf. [6], chapitre VII), so we omit details. Let $(T_n)_{n=1,2,\ldots}$ be a sequence of stopping times such that $T_n \nearrow \infty$, and $(M_{t\wedge T_n})_{t\in\mathbf{R}_+}$ are square-integrable martingales for $n = 1, 2, \ldots$ Considering the Doob-Meyer decompositions of the submartingales $\big((M^+_{t\wedge T_n})^2\big)_{t\in\mathbf{R}_+}$, we obtain a unique increasing, predictable process $A = (A_t)_{t\in\mathbf{R}_+}$, $A_0 = 0$, such that

(1.7) $\quad \big((M^+_{t\wedge T_n})^2 - A_{t\wedge T_n}\big)_{t\in\mathbf{R}_+}$ is a uniformly integrable martingale for $n = 1, 2, \ldots$

Now, to prove the first equality of (1.4) and (1.5) it is enough to show that

(1.8) $\quad \lim\limits_{t\to+\infty} M_t$ exists and is finite a.s. on $\{A_\infty < \infty\}$,

(1.9) $\quad \sup\limits_t M_t = +\infty$ a.s. on $\{A_\infty = +\infty\}$.

To prove (1.8), let $S = \inf\{t\colon A_t \geqslant a\}$, where a is any fixed positive number. It suffices to prove that $\lim\limits_{t\to+\infty} M_{t\wedge S}$ exists a.s. and is finite. This will be shown if we prove that

$$\sup_n \mathrm{E}\big((M^+_{T_n\wedge S})^2\big) < \infty, \tag{1.10}$$

and then apply Lemma 1 to the local martingale $(M_{t\wedge S})_{t\in\mathbf{R}_+}$. Now, S is a predictable stopping time, since it is the debut of a predictable set with sections "closed from the right" (cf., e.g., [1]); hence there exists a sequence $(S_m)_{m=1,2,\ldots}$ of stopping times such that $S_m \nearrow S$, $S_m < S$ for $m = 1, 2, \ldots$ Now, (1.7), the optional sampling theorem and the definition of S imply

$$\mathrm{E}\big((M^+_{T_n\wedge S_m})^2\big) = \mathrm{E}(A_{S_m\wedge T_n}) \leqslant a \quad \text{for } n, m = 1, 2, \ldots,$$

whence

$$a \geqslant \lim_{m\to+\infty} \mathrm{E}\big((M^+_{T_n\wedge S_m})^2\big) \geqslant \mathrm{E}\big((M^+_{S-})^2 \chi_{\{S\leqslant T_n\}}\big) + \mathrm{E}\big((M^+_{T_n})^2 \chi_{\{S>T_n\}}\big).$$

This, combined with the obvious inequality

$$(M^+_{S-})^2 \geqslant \frac{1}{2}(M^+_S)^2 - (\Delta M_S)^2,$$

gives

$$a \geqslant \frac{1}{2}\mathrm{E}\big((M^+_{T_n\wedge S})^2\big) - \mathrm{E}\big((\Delta M_S)^2 \chi_{\{S\leqslant T_n, S<\infty\}}\big)$$

and, finally,

$$\sup_n \mathrm{E}\big((M^+_{T_n\wedge S})^2\big) \leqslant 2a + 2\mathrm{E}\big((\Delta M_S)^2 \chi_{\{S<\infty\}}\big).$$

Thus, (1.10) is proved by using assumption (1.1).

To prove (1.9) we fix again an arbitrary positive number a and put $R = \inf\{t\colon M_t \geqslant a\}$. It suffices to show that $A_R < \infty$ a.s., and this follows from the inequality

$$\begin{aligned} M_{T_n\wedge R} &= M_{T_n}\chi_{\{T_n<R\}} + M_R\chi_{\{T_n\geqslant R\}} \\ &\leqslant a\chi_{\{T_n<R\}} + \Delta M_R\chi_{\{T_n\geqslant R, R<\infty\}} + M_{R-}\,\chi_{\{T_n\geqslant R\}} \leqslant a + |\Delta M_R|\chi_{\{R<\infty\}} \end{aligned}$$

and from the fact that

$$\mathrm{E}\big((M^+_{T_n\wedge R})^2\big) = \mathrm{E}(A_{T_n\wedge R}) \nearrow E(A_R).$$

Of course, assumption (1.1) is used here once more.

Finally, to prove the second equality of (1.4) we use the fact that $([M]_{t\wedge T_n} - \langle M\rangle_{t\wedge T_n})_{t\in \boldsymbol{R}_+}$ is a uniformly integrable martingale for $n = 1, 2, \ldots$ (cf. [5]) and that $\Delta[M]_t = (\Delta M_t)^2$. We argue analogously as in the first part of the proof. Details are omitted and the proof is completed.

Remarks. (a) It follows from the proof that if M is a locally square-integrable local martingale, then $[M]_\infty < \infty$ and $\lim\limits_{t\to+\infty} M_t$ exists and is finite a.s. on $\{\langle M\rangle_\infty < \infty\}$, whenever we require (1.1) to be satisfied for predictable stopping times only. For example, if the family $(\mathscr{F}_t)_{t\in\boldsymbol{R}_+}$ is quasi-left continuous, then $\Delta M_T = 0$ a.s. for each predictable stopping time (see [1], chap. V) and (1.1), in this modified form, is automatically satisfied.

(b) Assumption (1.1) is, of course, satisfied if the jumps of M are bounded. It is so, for example, for the local martingale associated with a point process (cf. [3]).

(c) The same theorem (without the statement concerning $[M]$) with the same proof holds if (1.1) is replaced by $\mathrm{E}(|\Delta M_T|^p \chi_{\{T<\infty\}}) < \infty$, and $\langle M\rangle$ by the increasing predictable process $A^{(p)}$ such that $|M|^p - A^{(p)}$ is a local martingale, where p is an arbitrary number not less than 1. In particular, considering $p = 1$, we get the following

COROLLARY. *If M is a local martingale ($M_0 = 0$) such that*

$$\mathrm{E}(|\Delta M_T|\chi_{\{T<\infty\}}) < \infty$$

for every stopping time T, then, for P-almost all ω, either $\lim\limits_{t\to+\infty} M_t(\omega)$ exists and is finite or

$$\limsup_{t\to+\infty} M_t(\omega) = +\infty = -\liminf_{t\to+\infty} M_t(\omega).$$

2. Let M be a locally square-integrable local martingale, $M_0 = 0$. We know that, under some assumptions, $(M_t)_{t\in\boldsymbol{R}_+}$ converges a.s. on the set $\{\langle M\rangle_\infty < \infty\}$ and diverges a.s. on $\{\langle M\rangle_\infty = +\infty\}$. In this section we show that in the most important cases the divergence cannot be too rapid. Namely, we have the following "continuous time version of the strong law of large numbers" (cf. [6], chapitre VII):

PROPOSITION 2. *Assume that the process $\langle M\rangle$ is continuous. Let $f\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ be any increasing continuous function such that*

$$\int_{\boldsymbol{R}_+} \frac{1}{1+f^2(x)}\,dx < \infty \tag{2.1}$$

and

$$\lim_{x\to+\infty} \frac{f(x+a)}{f(x)} = 1 \quad \textit{for each } a \geqslant 0. \tag{2.2}$$

Then

$$\lim_{t\to+\infty} \frac{M_t}{f(\langle M\rangle_t)} = 0 \ a.s. \quad \text{on the set } \{\langle M\rangle_\infty = +\infty\}. \tag{2.3}$$

This proposition should hold also without assumption (2.2) (f may, clearly, increase more rapidly!) but discussion of this condition does not seem to be interesting. For the "important" functions

$$f(x) = x^a \quad \text{and} \quad f(x) = \sqrt{x}(\log^+ x)^a \quad (a > \tfrac{1}{2})$$

condition (2.2) is satisfied.

The proof will be based on two lemmas.

LEMMA 2. *Let $(X_n, \mathcal{G}_n)_{n=0,1,\ldots}$ be a discrete time martingale such that* $\mathrm{E}(X_n^2) < \infty$ *for* $n = 1, 2, \ldots$ *and let*

$$\langle X\rangle_n = \sum_{k=1}^{n} \mathrm{E}\big((X_k - X_{k-1})^2 \,|\, \mathcal{G}_{k-1}\big) \quad \text{for } n > 0, \ \langle X\rangle_0 = 0.$$

Then

$$\left(\frac{X_n^2}{f^2(a+\langle X\rangle_n)} + \int\limits_{a+\langle X\rangle_n}^{+\infty} \frac{du}{f^2(u)}, \ \mathcal{G}_n\right)_{n=0,1,\ldots} \tag{2.4}$$

is a supermartingale for every function f which satisfies the assumptions of Proposition 2 and for each $a > 0$ such that $f(a) > 0$.

This lemma is known, and it can easily be proved by straightforward calculations, using the fact that $X^2 - \langle X\rangle$ is a martingale and $\langle X\rangle_n$ is measurable with respect to $\mathcal{G}_{n-1}$.

LEMMA 3. *If $N = (N_t)_{t\in R}$ is a square-integrable martingale* ($N_0 = 0$) *such that the process $\langle N\rangle$ is continuous, and f is as above, then*

$$\mathrm{P}\left(\sup_t \left(\frac{N_t^2}{f^2(a+\langle N\rangle_t)} + \int\limits_{a+\langle N\rangle_t}^{+\infty} \frac{du}{f^2(u)}\right) > r\right) \leqslant \frac{1}{r}\int\limits_a^{+\infty} \frac{du}{f^2(u)}$$

for every $r > 0$, and a such that $f(a) > 0$.

(Notice that the process under the sup may not, in general, be a supermartingale.)

Proof. By the right continuity of N, $\langle N\rangle$ and f, it suffices to prove that

$$\mathrm{P}\left(\max_{0\leqslant k\leqslant m} \left(\frac{N_{k2^{-n}}^2}{f^2(a+\langle N\rangle_{k2^{-n}})} + \int\limits_{a+\langle N\rangle_{k2^{-n}}}^{+\infty} \frac{du}{f^2(u)}\right) > r\right) \leqslant \frac{1}{r}\int\limits_a^{+\infty} \frac{du}{f^2(u)} \tag{2.5}$$

for $n, m = 1, 2, \ldots$

Fix n, m, and set

$$X_j^{(\nu)} = N_{j2^{-(n+\nu)}}, \qquad \mathcal{G}_j^{(\nu)} = \mathcal{F}_{j2^{-(n+\nu)}},$$

$$\langle X^{(\nu)}\rangle_j = \sum_{i=1}^{j} \mathrm{E}\big((X_i^{(\nu)} - X_{i-1}^{(\nu)})^2 \,|\, \mathcal{G}_{i-1}^{(\nu)}\big), \tag{2.6}$$

where ν is an arbitrary integer. Now, $(X_j^{(\nu)}, \mathcal{G}_j^{(\nu)})_{j=0,1,\ldots}$ is a martingale and we may apply Lemma 2. We write "the maximal inequality" for the (non-negative) supermartingale (2.4):

$$\mathrm{P}\left(\max_{0\leqslant j\leqslant m2^{\nu}}\left(\frac{(X_j^{(\nu)})^2}{f^2(a+\langle X^{(\nu)}\rangle_j)} + \int\limits_{a+\langle X^{(\nu)}\rangle_j} \frac{du}{f^2(u)}\right) > r\right) \leqslant \frac{1}{r}\int\limits_a^{+\infty} \frac{du}{f^2(u)}.$$

Hence

$$\mathrm{P}\left(\max_{0\leqslant k\leqslant m}\left(\frac{N_{k2^{-n}}^2}{f^2(a+\langle X^{(\nu)}\rangle_{k2^{\nu}})} + \int\limits_{a+\langle X^{(\nu)}\rangle_{k2^{\nu}}} \frac{du}{f^2(u)}\right) > r\right) \leqslant \frac{1}{r}\int\limits_a^{+\infty} \frac{du}{f^2(u)}. \tag{2.7}$$

It is known [2] that

$$\lim_{\nu\to+\infty} \langle X^{(\nu)}\rangle_{k2^{\nu}} = \langle N\rangle_{k2^{-n}} \quad \text{in } L^1,$$

since the process $\langle N\rangle$ is continuous. Therefore, there exists a subsequence (ν_i) such that

$$\langle X^{(\nu_i)}\rangle_{k2^{\nu_i}} \to \langle N\rangle_{k2^{-n}} \quad \text{as } i\to+\infty$$

a.s. for $k = 0, 1, \ldots, m$. Inequality (2.5) is now obtained from (2.7) by passing to the limit.

Proof of Proposition 2. Let $(T_n)_{n=1,2,\ldots}$ be a sequence of stopping times ($T_n \nearrow \infty$) such that $(M_{t\wedge T_n})_{t\in R_+}$ is a square-integrable martingale for $n = 1, 2, \ldots$ From Lemma 3, applied to the martingale $(M_{t\wedge T_n})_{t\in R_+}$, we derive immediately that

$$\mathrm{P}\left(\sup_t \frac{|M_{t\wedge T_n}|}{f(a+\langle M\rangle_{t\wedge T_n})} > \sqrt{r}\right) \leqslant \frac{1}{r}\int\limits_a^{+\infty} \frac{du}{f^2(u)}$$

for $n = 1, 2, \ldots$, $r > 0$, and for u such that $f(u) > 0$. Hence, letting $n \to \infty$ we get

$$\mathrm{P}\left(\sup_t \frac{|M_t|}{f(a+\langle M\rangle_t)} > \sqrt{r}\right) \leqslant \frac{1}{r}\int\limits_a^{+\infty} \frac{du}{f^2(u)}.$$

Let $(a_n)_{n=1,2,\ldots}$ be a sequence such that

$$\sum_{n=1}^{\infty} \int\limits_{a_n}^{+\infty} \frac{du}{f^2(u)} < \infty.$$

By the Borel-Cantelli lemma we have

$$\sup_t \frac{|M_t|}{f(a_n + \langle M \rangle_t)} \leqslant \sqrt{r} \quad \text{for } n \text{ large a.s.}$$

Now, fix ω for which this inequality is satisfied, and such that $\langle M \rangle_\infty(\omega) = +\infty$. For $n = n(\omega)$, sufficiently large but fixed, it follows that

$$\frac{|M_t(\omega)|}{f(\langle M \rangle_t(\omega))} \leqslant \sqrt{r}\, \frac{f(a_n + \langle M \rangle_t(\omega))}{f(\langle M \rangle_t(\omega))} \quad \text{for } t \geqslant 0.$$

Finally, using assumption (2.2) (this is the only place where we make use of it), and letting $r \to 0$ over a countable set, we obtain (2.3).

Added in proof. Similar topics are considered in the paper by D. Lepingle, *Sur le comportement asymptotique des martingales locales*, Séminaire de Probabilités XII, Lecture Notes in Mathematics 649 (1978), p. 148-161. It has turned out that some of our assumptions can be weakened.

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