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# ON TRANSLATION INVARIANT FAMILIES OF SETS

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In [3] it has been shown how several analogies between topological and measure spaces can be subsumed under an abstract theory of Baire category. There are, however, additional analogies concerning Hamel bases, translations of sets, and decompositions of sets. The purpose of this note is to indicate how some of these results can be unified.

Unless otherwise stated, $X$ denotes the real line with the usual topology and algebraic operations. We shall also use the terminology and theorems of [3] (cf. also [4]).

Definition. A family $\mathscr{C}$ of subsets of a fixed non-empty set $X$ is called a $\mathfrak{K}$-*family* if it satisfies the following axioms:

(a) $X = \bigcup \mathscr{C}$.

(b) Let $A$ be a $\mathscr{C}$-set and let $\mathscr{D}$ be a non-empty family of disjoint $\mathscr{C}$-sets which has power less than the power of $\mathscr{C}$. If $A \cap (\bigcup \mathscr{D})$ contains a $\mathscr{C}$-set, then there is a $\mathscr{D}$-set $D$ such that $A \cap D$ contains a $\mathscr{C}$-set. On the other hand, if $A \cap (\bigcup \mathscr{D})$ contains no $\mathscr{C}$-set, then there is a $\mathscr{C}$-set $B \subset A$ which is disjoint from all $\mathscr{D}$-sets.

Notation. If $S$ is a subset of $X$ and $t \in X$, then $S(t) = \{x+t\colon x \in S\}$.

Definition. A $\mathfrak{K}$-family $\mathscr{C}$ is called an $\mathfrak{S}$-*family* if it satisfies the following conditions:

(1) $\mathscr{C}$ is translation invariant.

(2) If $A$ is a $\mathscr{C}$-set and $D$ is a topologically dense subset of $X$, then $\bigcup_{t \in D} A(t)$ is a $\mathscr{C}_{\mathrm{II}}$-set everywhere.

Remark 1. It follows immediately from condition (2) that every $\mathscr{C}$-set is a $\mathscr{C}_{\mathrm{II}}$-set.

Here are three examples of $\mathfrak{S}$-families.

Example 1. Let $\mathscr{C}$ be the family of all (non-empty) open intervals. The sets which have the Baire property with respect to $\mathscr{C}$ are the sets with the classical Baire property.

Example 2. Let $\mathscr{C}$ be the family of all compact sets of positive Borel measure. The sets which have the Baire property with respect to $\mathscr{C}$ are the Lebesgue measurable sets.

Example 3. Let $\mathscr{I}$ be a translation invariant (proper) $\sigma$-ideal and let $\mathscr{C} = \{A: X \sim A \in \mathscr{I}\}$. The family of sets having the Baire property with respect to $\mathscr{C}$ is $\mathscr{C} \cup \mathscr{I}$.

Throughout this paper $\mathscr{C}$ will denote an arbitrary $\mathfrak{S}$-family and the Baire property will be with respect to $\mathscr{C}$.

Condition (1) immediately yields the following

THEOREM 1. *The families of $\mathscr{C}$-singular sets, $\mathscr{C}_{\mathrm{I}}$-sets, and sets with the Baire property are translation invariant.*

Upon the application of the Fundamental Theorem ([3], Theorem 2) a strengthening of condition (2) is obtained.

THEOREM 2. *If $S$ is a $\mathscr{C}_{\mathrm{II}}$-set and $D$ is a topologically dense subset of $X$, then $\bigcup_{t\in D} S(t)$ is a $\mathscr{C}_{\mathrm{II}}$-set everywhere.*

Proof. Let $S$ be a $\mathscr{C}_{\mathrm{II}}$-set everywhere on a $\mathscr{C}$-set $A$ and let $E$ be a countable, topologically dense subset of $D$. Suppose $B$ is any $\mathscr{C}$-set. From condition (2), $B \cap \left[\bigcup_{t\in E} A(t)\right]$ is a $\mathscr{C}_{\mathrm{II}}$-set. Hence, for some $t \in E$, $B \cap A(t)$ is a $\mathscr{C}_{\mathrm{II}}$-set, whence, in view of Theorem 1 of [3], it contains a $\mathscr{C}$-set $C$. Since $S(t)$ is a $\mathscr{C}_{\mathrm{II}}$-set everywhere on $A(t)$, $C \cap S(t)$ is a $\mathscr{C}_{\mathrm{II}}$-set. Therefore, $B \cap \left[\bigcup_{t\in D} S(t)\right]$ is a $\mathscr{C}_{\mathrm{II}}$-set for every $\mathscr{C}$-set $B$.

COROLLARY. *If $S$ is a $\mathscr{C}_{\mathrm{II}}$-set with the Baire property and $D$ is a topologically dense subset of $X$, then $\bigcup_{t\in D} S(t)$ is a $\mathscr{C}$-residual set.*

The following theorem generalizes a result found in [11]:

THEOREM 3. *If $D$ is a topologically dense subset of $X$, $S$ and $T$ are $\mathscr{C}_{\mathrm{II}}$-sets, and $S$ has the Baire property, then there is a denumerable subset $D_0$ of $D$ such that $S \cap T(a)$ is a $\mathscr{C}_{\mathrm{II}}$-set for every $a \in D_0$.*

Proof. Let $S$ be a $\mathscr{C}_{\mathrm{II}}$-set everywhere on a $\mathscr{C}$-set $A$, let $E$ be a countable, topologically dense subset of $D$, and let

$$U = \bigcup_{a\in E} T(a).$$

Since $U$ is a $\mathscr{C}_{\mathrm{II}}$-set everywhere, $U \cap A$ is a $\mathscr{C}_{\mathrm{II}}$-set. From the assumption that $S$ has the Baire property it follows that $A \sim S$ is a $\mathscr{C}_{\mathrm{I}}$-set. Hence $U \cap (A \cap S)$ and $U \cap S$ are $\mathscr{C}_{\mathrm{II}}$-sets. There exists then an element $a_1 \in E$ such that $S \cap T(a_1)$ is a $\mathscr{C}_{\mathrm{II}}$-set. Having defined $a_1, \ldots, a_n$, repeat the reasoning to obtain an element $a_{n+1} \in E \sim \{a_1, \ldots, a_n\}$ such that $S \cap T(a_{n+1})$ is a $\mathscr{C}_{\mathrm{II}}$-set. Let $D_0 = \{a_1, a_2, \ldots\}$.

The category version of the following theorem is due to Banach [2].

THEOREM 4. *If $G$ is a topologically dense, additive subgroup of $X$ which has the Baire property with respect to $\mathscr{C}$, then $G$ is either a $\mathscr{C}_{\mathrm{I}}$-set or identical to $X$.*

Proof. Assume $G$ is a $\mathscr{C}_{\mathrm{II}}$-set and let $D$ be a topologically dense subset of $G$. Suppose $x$ is any element of $X$. Since $G$ has the Baire property, it follows, as in the proof of Theorem 3, that

$$G\cap\Big[\bigcup_{t\in D} G(t-x)\Big]$$

is a $\mathscr{C}_{\mathrm{II}}$-set. Hence there is an index $t$ such that $G\cap G(t-x)\neq 0$. Selecting an element $y\in G\cap G(t-x)$, we have $x+y=z+t$ for some $z\in G$. Therefore $x=(z+t)-y\in G$.

Remark 2. Since every uncountable, additive subgroup of $X$ is topologically dense, one can replace the hypothesis that $G$ is topologically dense by the hypothesis that every countable set is a $\mathscr{C}_{\mathrm{I}}$-set.

Remark 3. Theorem 4 is also true when $X$ is the set of all non-zero real numbers with the usual relativized topology, the group operation is multiplication, and $\mathscr{C}$ consists of all (non-empty) open intervals contained in $X$; as is easily seen from the proof of Theorem 4.

The existence of sets which do not have the Baire property is obtained from the well-known construction of Vitali. First, we prove a basic lemma.

LEMMA. *If $S$ is a subset of $X$, $D$ is a topologically dense set, and*

$$S(r)\cap S(t)=0 \quad \text{for all } r,t\in D,\ r\neq t,$$

*then the complement of $S$ is a $\mathscr{C}_{\mathrm{II}}$-set everywhere (or, equivalently, $S$ contains no $\mathscr{C}_{\mathrm{II}}$-set with the Baire property).*

Proof. The conclusion is obvious if $S$ is a $\mathscr{C}_{\mathrm{I}}$-set. Assume $S$ is a $\mathscr{C}_{\mathrm{II}}$-set, let $a$ be a fixed element of $D$, and let $E=D\sim\{a\}$. From Theorem 2 and the inclusion

$$\bigcup_{t\in E} S(t)\subset X\sim S(a)$$

it follows that $X\sim S(a)$, and hence also $X\sim S$, is a $\mathscr{C}_{\mathrm{II}}$-set everywhere.

THEOREM 5. *$X$ can be decomposed into denumerably many, disjoint, congruent sets, none of which has the Baire property with respect to any $\mathfrak{S}$-family.*

Proof. Let $Q$ denote the set of rational numbers. Define an equivalence relation on $X$ by $x\equiv y$ if $x-y\in Q$. Choose one member from each distinct equivalence class and let $S$ be the set of elements selected. Then

$$X=\bigcup_{t\in Q} S(t) \quad \text{and} \quad S(r)\cap S(t)=0 \quad \text{for } r,t\in Q,\ r\neq t.$$

THEOREM 6 (cf. [5]). *Every $\mathscr{C}_{II}$-set with the Baire property contains a set which does not have the Baire property.*

Proof. Let $S$ be a $\mathscr{C}_{II}$-set with the Baire property, let $D$ be a countable, topologically dense subset of $X$, let

$$U = \bigcup_{t \in D} S(t),$$

and let $N$ be a set which does not have the Baire property. The equality $N = (N \cap U) \cup (N \sim U)$ and the fact that $X \sim U$ is a $\mathscr{C}_I$-set imply the existence of an index $t \in D$ such that $N \cap S(t)$ does not have the Baire property. Hence $S \cap N(-t)$ does not have the Baire property.

In [7] Sierpiński proved that a Hamel basis has inner Lebesgue measure zero and in [10] he proved the category analogue that the complement of a Hamel basis is everywhere of the second category. In the case of measure, Ruziewicz proved a stronger result.

THEOREM 7 (cf. [6]). *If $B$ is a Hamel basis and $U$ is a union of fewer than continuum many translates of $B$, then the complement of $U$ is a $\mathscr{C}_{II}$-set everywhere.*

Proof. Let $I$ be a set of real numbers of power less than the power of the continuum, let

$$U = \bigcup_{t \in I} B(t),$$

let $D$ be a topologically dense set of non-zero rational numbers such that $|r-s| \neq 1$ for all $r, s \in D$, and let $b$ be an element of $B$ which does not occur in the Hamel expansion (in base $B$) of any element of $I$.

Assume $r, s \in D$ and $U(br) \cap U(bs) \neq 0$; then there exist $t_1, t_2 \in I$ and $b_1, b_2 \in B$ such that $b_1 + t_1 + br = b_2 + t_2 + bs$. Since the expansions of $t_1$ and $t_2$ do not contain the element $b$, it is easily seen that in the expansion of $b_1 + t_1 + br$ the element $b$ occurs either with coefficient $r$ or with $1+r$ and in the expansion of $b_2 + t_2 + bs$ the element $b$ occurs either with coefficient $s$ or with $1+s$. From the uniqueness of Hamel expansions, one of the following equalities holds:

$$r = s, \quad r = 1+s, \quad 1+r = s, \quad 1+r = 1+s.$$

By virtue of the choice of $D$, we must have $r = s$. The conclusion now follows from the Lemma.

COROLLARY. *A Hamel basis with the Baire property is a $\mathscr{C}_I$-set.*

THEOREM 8 (cf. [1] and [7]). *The set $S$ of real numbers whose representation with respect to a given Hamel basis $B$ does not contain a given fixed element $b \in B$ does not have the Baire property. In fact, both $S$ and $X \sim S$ are $\mathscr{C}_{II}$-sets everywhere.*

Proof. Let $Q$ denote the set of all rational numbers, let $c$ be a fixed element of $B \sim \{b\}$, and let $T$ be the set of all real numbers representable as a rational linear combination of elements of $B \sim \{b, c\}$. From the equalities

$$X = \bigcup_{r \in Q} S(br) \quad \text{and} \quad S = \bigcup_{r \in Q} T(cr)$$

it is seen that $S$ and $T$ are $\mathscr{C}_{II}$-sets. By Theorem 2 and the Lemma, $S$ and $X \sim S$ are $\mathscr{C}_{II}$-sets everywhere.

Finally, we unify two theorems regarding the functional equation $f(x+y) = f(x)+f(y)$.

Definition. A function $f\colon X \to X$ has the *Baire property with respect to* $\mathscr{C}$ if the inverse image of each open set has the Baire property with respect to $\mathscr{C}$.

THEOREM 9 (cf. [8] and [9]). *Suppose every Borel set has the Baire property with respect to $\mathscr{C}$ and $\mathscr{C}$ is invariant under reflection about 0. If $f(x)$ has the Baire property and satisfies the functional equation $f(x+y) = f(x)+f(y)$ for all $x, y \in X$, then $f(x) = ax$ for all $x$, where $a = f(1)$.*

Proof. Set $\varphi(x) = f(x) - x \cdot f(1)$; then $\varphi(x+y) = \varphi(x)+\varphi(y)$ for all $x, y$. It can be shown that, for every rational number $r$, $\varphi(r) = 0$. Hence

$$(*) \qquad \varphi(x+r) = \varphi(x) \quad \text{for every rational number } r.$$

Let $E = \{x\colon \varphi(x) = 0\}$, $F = \{x\colon \varphi(x) > 0\}$, and $G = \{x\colon \varphi(x) < 0\}$. Since $\varphi(-x) = -\varphi(x)$, the sets $F$ and $G$ are symmetric about zero. Assume, for some real number $b$, that $\varphi(b) \neq 0$.

We first show $F$ is a $\mathscr{C}_{II}$-set. This is clear if $E$ is a $\mathscr{C}_{I}$-set. On the other hand, if $E$ is a $\mathscr{C}_{II}$-set, then so also is

$$E(-b) = \{x-b\colon \varphi(x) = 0\} = \{y\colon \varphi(y+b) = 0\}.$$

But if $\varphi(y+b) = 0$, then, since $\varphi(y+b) = \varphi(y)+\varphi(b)$, $y$ does not belong to $E$. Hence $E(-b)$ is disjoint from $E$. Thus, in any case, $F$ is a $\mathscr{C}_{II}$-set.

It follows from $(*)$ that $F$ and $G$ are $\mathscr{C}_{II}$-sets everywhere; whence $F$ and $X \sim F$ are $\mathscr{C}_{II}$-sets everywhere, and $F$ does not have the Baire property. However, this contradicts the fact that $\varphi(x)$ has the Baire property. Therefore, $\varphi(x) = 0$ for all real numbers $x$.

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