COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

UNIFORMLY CONTINUOUS FUNCTIONALS ON BANACH ALGEBRAS

BY

ANTHONY TO-MING LAU[1] (EDMONTON)

1. Introduction. In [9], Granirer defined the subspace of uniformly continuous functionals on the Fourier algebra $A(G)$ of a locally compact group G, denoted by $UC(\hat{G})$, to be the norm closure of $A(G)\cdot VN(G)$, where $VN(G)$ is the von Neumann algebra generated by the left translation operators on $L_2(G)$. In this case, $UC(\hat{G})$ is always a C^*-algebra (see [10, p. 65]). Also $UC(\hat{G})$ is precisely the space of bounded uniformly continuous complex-valued function on $\hat{G}$ (the dual group of G) when G is abelian.

By an *F-algebra* we shall mean a pair (A, M) such that A is a complex Banach algebra and M is a von Neumann algebra such that $A = M_*$, the predual of M, and the identity of M is a multiplicative linear functional on A. If there is no confusion, we shall simply say that A is an F-algebra and we shall identify A^* with M. The class of F-algebras was introduced and studied by the author in [15]. It includes the algebra $L_1(G)$ and $A(G)$ of a locally compact group G. It also includes the Fourier–Stieltjes algebra $B(G)$ of G, the measure algebra of a locally compact semigroup or hypergroups [5], the class of convolution measure algebras studied by Taylor [22] and the class of L-algebras (for which the identity of the dual algebra is in the spectrum of the L-algebra) considered by McKilligan and White [19].

In this paper, we define and study the spaces $UC_r(A)$ and $UC_l(A)$ of right and left uniformly continuous functionals on F-algebras. We prove (Corollary 4.5) among other things that if A is an F-algebra with a right approximate identity bounded by one and X is an ultra-weakly dense C^*-subalgebra of A^* which is topologically invariant topologically left introverted contained in $UC_r(A)$, then the algebra of bounded right multipliers of A is isometric and anti-isomorphic to the largest closed subalgebra of X^* (with the Arens product) containing A as a right ideal. We also prove (Theorem 5.1) that if A is an F-algebra with a right approximate

[1] This research is supported by NSERC Grant A 7679.

identity bounded by one, then $UC_r(A)^*$ is isometric and algebra isomorphic to the algebra of bounded operators from A^* into A^* commuting with the action of A.

2. Preliminaries and some notations. Let E be a linear space, and ϕ be a linear functional on E; then the value of ϕ at an element x in E will be written as $\phi(x)$ or $\langle \phi, x \rangle$. If F is a subspace of the algebraic dual of E, then $\sigma(E, F)$ will denote the weakest locally convex topology on E such that each of the functionals in F is continuous.

If K is a subset of a normed linear space E, then $\langle K \rangle$ will denote the linear span of K. Also the closure of K and the closed linear span of K will be denoted by $\bar{K}$ and $\langle K \rangle^-$ respectively when the closure is taken with respect to the norm topology, or by $\bar{K}^\tau$ and $\langle K \rangle^{-\tau}$ when the closure is taken with respect to a topology τ on E different from the norm topology. The continuous dual of E will be denoted by E^*.

If M is a von Neumann algebra, then M_* will denote its unique predual. The topology $\sigma(M, M_*)$ on M will be referred to as the ultraweak, or simply the σ-topology. A linear functional m in M^* is called a *state* if $\|m\| = m(e) = 1$, where e is the identity in M. The following lemma is probably in the folklore. However we are unable to find a reference to it.

LEMMA 2.1. *Let M be a von Neumann algebra. Then the set of states in M_* is weak*-dense in the set of states in M^*.*

Proof. Let m be a state in M^*. By Goldstine's theorem [4, p. 424], there exists a net $\psi_\alpha \in M_*$, $\|\psi_\alpha\| \leqslant 1$, such that ψ_α converges to m in the weak*-topology. Since $|\psi_\alpha(e)| \to m(e) = 1$, it follows that $\|\psi_\alpha\| \to \|m\|$. Let $\phi_\alpha = |\psi_\alpha|$ absolute value of ψ_α in M_*. It follows from [7, Lemma 3.3] that ϕ_α is also the restriction of the absolute value of ψ_α in M^* to M. Hence by [7, Lemma 3.5], ϕ_α converges to m in the weak*-topology of M^*. Consequently $\phi_\alpha/\phi_\alpha(e)$ are states in M_* converging to m in the weak*-topology.

If A is a Banach algebra, then for each $\phi \in A$, and $x \in A^*$, define the elements $\phi \cdot x$ and $x \cdot \phi$ in A^* by

$$\langle x \cdot \phi, \gamma \rangle = \langle x, \phi \cdot \gamma \rangle \quad \text{and} \quad \langle \phi \cdot x, \gamma \rangle = \langle x, \gamma \cdot \phi \rangle$$

for each $\gamma \in A$. We say that a subspace X of A^* is *topologically left* (resp. *right*) *invariant* if $X \cdot \phi \subseteq X$ (resp. $\phi \cdot X \subseteq X$) for each $\phi \in A$; X is *topologically invariant* if it is both left and right topologically invariant.

If X is a topologically left invariant subspace of A^* and $m \in X^*$, we define an operator m_L from X into A^* by

$$\langle m_L(x), \phi \rangle = \langle x \cdot \phi, m \rangle \quad \text{for each } \phi \in A.$$

We say that X is *topologically left introverted* if $m_L(X) \subseteq X$ for each m in X^*. Similarly, we can define topologically right introverted subspaces of A^*.

A subspace X of A^* is *topologically introverted* if it is both left and right topologically introverted.

In [1], Arens defined a product on the second conjugate space A^{**} by

$$\langle m \odot n, x\rangle = \langle m, n_L(x)\rangle \quad \text{for each } m, n \in A^{**}, x \in A^*.$$

Then A^{**} with respect to this product becomes a Banach algebra. If X is a topologically left invariant and left introverted subspace of A^*, then the Arens product on X^* makes sense and renders X^* into a Banach algebra.

If A is an F-algebra, let $P_1(A)$ denote all positive functionals ϕ in A such that $\phi(e) = 1$. In this case, $\phi \cdot \psi \in P_1(A)$ whenever $\phi, \psi \in P_1(A)$.

LEMMA 2.2. *Let A be an F-algebra and X be a topologically left invariant subspace of A^*. Then X is topologically left introverted if and only if for each x in X the ultraweak closure of the set $K(x) = \{\phi \cdot x;\ \phi \in P_1(A)\}$ is contained in X. In particular, any ultraweakly closed topologically invariant subspace of A^* is topologically introverted.*

Proof. If X is topologically introverted, and $y \in \overline{K(x)^\sigma}$, $x \in X$, then there exists a net $\{\phi_\alpha\}$ in $P_1(A)$ such that $\phi_\alpha \cdot x$ converges to y in the ultraweak topology. Let m be a weak*-cluster point of $\{\phi_\alpha\}$ in A^{**}. Then for each $\gamma \in A$,

$$\langle y, \gamma\rangle = \lim_\alpha \langle \phi_\alpha \cdot x, \gamma\rangle = \lim_\alpha \langle x \cdot \gamma, \phi_\alpha\rangle = \langle x \cdot \gamma, m\rangle = \langle m_L(x), \gamma\rangle.$$

Hence $y = m_L(x) \in X$. Conversely if $\overline{K(x)^\sigma} \subseteq X$ for each $x \in X$, let m be a state on A^*, i.e. $m \geqslant 0$ and $m(e) = 1$. By Lemma 2.1, there exists a net $\{\phi_\alpha\}$ in $P_1(A)$ converging to m in the weak*-topology of A^{**}. Then if $\gamma \in A$,

$$\langle m_L(x), \gamma\rangle = \langle m, x \cdot \gamma\rangle = \lim_\alpha \langle \phi_\alpha, x \cdot \gamma\rangle = \lim_\alpha \langle \phi_\alpha \cdot x, \gamma\rangle.$$

Hence $m_L(x) \in \overline{K(x)^\sigma} \subseteq X$. Since every functional in X^* is extendable to a functional on A^* and each element in A^* is the linear combination of states, it follows that $m_L(x) \in X$ for each $m \in X^*$ and each $x \in X$.

If G is a locally compact group with a fixed left Haar measure λ, let $L_p(G)$ $(1 \leqslant p < \infty)$ denote the Banach space of complex-valued measurable functions f on G such that $|f|^p$ is integrable. The Banach algebras $L_1(G)$ and $M(G)$ are as defined in [12]. We also refer the readers to [8] the definitions and properties of the algebras $A(G)$, $B(G)$ and $VN(G)$.

3. Uniformly continuous functionals. Let A be a Banach algebra. Denote by

$$UC_r(A) = \langle A^* \cdot A\rangle^- \qquad \text{and} \qquad UC_l(A) = \langle A \cdot A^*\rangle^-.$$

Elements in $UC_r(A)$ (resp. $UC_l(S)$) are called *right* (resp. *left*) *uniformly continuous functionals* on A. Also, elements in $UC(A) = UC_r(A) \cap UC_l(A)$ are called *uniformly continuous functionals.*

If $A = L_1(G)$ of a locally compact group G, then $UC_r(A) = UC_r(G)$, and $UC_l(A) = UC_l(G)$ the spaces of bounded right and left uniformly continuous functions on G respectively (see [12, 32.45]). Also if A has a bounded right (resp. left) approximate identity, then the Cohen's factorization theorem [12, 32.22] implies that $UC_r(A) = A^* \cdot A$ (resp. $UC_l(A) = A \cdot A^*$).

PROPOSITION 3.1. *Let A be a Banach algebra. Then $UC_r(A)$ is a topologically invariant, topologically left introverted subspace of A^*. If A is an F-algebra, then $UC_r(A)$ is also invariant under the involution of A^*.*

Proof. It is clear the $UC_r(A)$ is topologically invariant. To see that it is topologically left introverted, let $m \in UC_r(A)^*$, $y = x \cdot \phi$, then $m_L(y) = m_L(x) \cdot \phi$. Hence $m_L(y) \in UC_r(A)$. Consequently, $m_L(UC_r(A)) \subseteq UC_r(A)$ by continuity of the operator m_L.

Finally if A is an F-algebra, $\phi \in A$, $\phi \geqslant 0$, $x \in A^*$ and $y = x \cdot \phi$, then for any $\gamma \in A$, $\gamma \geqslant 0$, $\phi \cdot \gamma \geqslant 0$. Hence $\langle y^*, \gamma \rangle = \langle x, \phi \cdot \gamma \rangle = \langle x^*, \phi \cdot \gamma \rangle = \langle x^* \cdot \phi, \gamma \rangle$, i.e. $y^* = x^* \cdot \phi$. Consequently $y^* \in UC_r(A)$ whenever $y \in UC_r(A)$ by continuity of involution.

An element $x \in A^*$ is *almost periodic* (resp. *weakly almost periodic*) if the map $\phi \to x \cdot \phi$ from A into A^* is a compact (resp. weakly compact) operator. Let $AP(A)$ and $WAP(A)$ denote the collection of almost periodic functionals and weakly almost periodic functionals on A respectively. Then as known, $x \in AP(A)$ (resp. $x \in WAP(A)$) if and only if the map $\phi \to \phi \cdot x$ from A into A^* is compact (resp. weakly compact) (see [21, Theorem 4.3] and [18, Theorem 2.2]). Let $R(A)$ denote the closed linear span of the spectrum of A.

PROPOSITION 3.2. *If A is an F-algebra, then each of the spaces $R(A)$, $AP(A)$ and $WAP(A)$ is closed, topologically invariant, topologically introverted and invariant under the involution of A^*. Furthermore $R(A) \subseteq UC(A)$ and $\langle e \rangle \subseteq R(A) \subseteq AP(A) \subseteq WAP(A)$.*

Proof. The first statement follows from [4, pp. 482–487]. If x is in the spectrum of A, let $\phi \in A$ such that $\phi(x) = 1$. Then $x \cdot \phi = \phi \cdot x = x$. Hence $R(A) \subseteq UC(A)$.

To see that $WAP(A)$ is topologically introverted, we simply note that if $x \in WAP(A)$, the set $\overline{K(x)}$ is weakly compact, where $K(x)$ is as defined in Lemma 2.2. Hence the weak and the ultraweak topologies agree on $\overline{K(x)}$. Consequently, $\overline{K(x)} = \overline{K(x)}^{\sigma} \subseteq WAP(A)$. Proofs of the other cases are similar.

PROPOSITION 3.3. *Let A be a Banach algebra.*

(a) *If A has a left approximate identity, then $UC_r(A)$ is weak* dense in A^*.*

(b) *If A has a bounded left approximate identity, then $UC_r(A)$ contains $WAP(A)$.*

Proof. (a) Let $\{\phi_\alpha\}$ be a left approximate identity for A, and let $\psi \in A$

such that $\psi(x) = 0$ for all $x \in UC_r(A)$. Then for each $x \in A^*$, we have

$$\langle x, \psi \rangle = \lim \langle x, \phi_\alpha \cdot \psi \rangle = \lim \langle x \cdot \phi_\alpha, \psi \rangle = 0.$$

Hence $\psi = 0$. Consequently $UC_r(A)$ is weak* dense in A^*.

(b) Let $x \in WAP(A)$ and $\{\phi_\alpha\}$ be a bounded left approximate identity for A, then, by passing to a subnet if necessary, we may assume that the net $\{x \cdot \phi_\alpha\}$ converges to some y in A^* in the weak topology. On the other hand, the net $\{x \cdot \phi_\alpha\}$ converges to x in the weak* topology of A^*. Hence $y = x$. Consequently $x \in UC_r(A)$.

Proposition 3.3 (b) is due to Granirer [10, Proposition 1] when $A = A(G)$ of an amenable G.

COROLLARY 3.4. *If A is a C^*-algebra, then $A^* = UC(A) = WAP(A)$.*

Proof. In this case, A always has a bounded approximate identity. Hence $UC(A) \supseteq WAP(A)$ by Proposition 3.3. However, it follows from [2, Theorem 7.1] that $A^* = WAP(A)$. The assertion follows.

Remark 3.5. Both Proposition 3.3 (a) and (b) are false when A does not have a left approximate identity. Indeed if M is any von Neumann algebra with dimension more than one, and $A = M_*$. Define on A a multiplication $\phi \cdot \psi = \psi(e)\phi$, $\phi, \psi \in A$. Then A is an F-algebra, and A cannot have a left approximate identity. Also $WAP(A) = A^*$ but $UC(A) = UC_r(A) = \{\lambda e;\ \lambda \in C\}$.

4. **Multipliers.** Let G be a locally compact group. Wendel in [23] has identified the algebra of multipliers of $L_1(G)$ with the measure algebra $M(G)$. McKennon [17, Theorem 5.7] proved an analogue of Wendel's result for the Fourier algebra $A(G)$ of a locally compact amenable group. He proved in this case that the multipliers of $A(G)$ can be identified with the Fourier–Stieltjes algebra $B(G)$. We shall prove in this section a generalization of Wendel and McKennon's results. We first establish a technical lemma that we shall need.

LEMMA 4.1. *Let A be an F-algebra. If A has a right approximate identity bounded by one, then A has a right approximate identity consisting of states.*

Proof. Let m be a weak*-cluster point in A^{**} of a right approximate identity in A bounded by one. Then m is a right identity in A^{**} and $\|m\| \leqslant 1$. Also $m(e) = 1$. Hence m is a state on A^*. By Lemma 2.1, there exists a net $\{\phi_\alpha\}$ in $P_1(A)$ converging to m in the weak*-topology. Then $\{\phi_\alpha\}$ is a weak right approximate identity of A consisting of states. Now an argument similar to that in the proof of [20, Theorem 2.2] shows that A also has a right approximate identity consisting of states.

For the rest of this section, we shall assume that A is an F-algebra and X is a topologically invariant and topologically left introverted closed subspace of A^* containing an ultraweakly dense C^*-subalgebra of A^*. Let r: $A \to X^*$ be the restriction map. Then an application of the Kaplansky density theorem shows that r is a linear isometry and an algebra homomorphism from A into a norm closed subalgebra of X^*. We shall identify ϕ with $r(\phi)$ and write

$$J_R(X^*) = \{m \in X^*;\ \phi \odot m \in A \text{ for all } \phi \in A\}.$$

Then clearly $J_R(X^*)$ is a closed subalgebra of X^* containing A as a right ideal. Furthermore if L is any subalgebra of X^* containing A as a right ideal, then $L \subseteq J_R(X^*)$.

Let $M_R(A)$ denote the algebra of bounded right multipliers T of A, i.e. T is a bounded linear operator from A into A such that $T(\phi \cdot \psi) = \phi \cdot T(\psi)$ for each $\phi, \psi \in A$. If $m \in J_R(X^*)$, let $U(m)$: $A \to A$ be defined by $U(m)(\phi) = \phi \odot m$. Then clearly $U(m) \in M_R(A)$ for each $m \in J_R(X^*)$. We list below a few properties of the mapping U: $m \to U(m)$ from $J_R(X^*)$ into $M_R(A)$.

PROPOSITION 4.2. (a) *U is a linear norm-decreasing algebra anti-homomorphism from $J_R(X^*)$ into $M_R(A)$.*

(b) *If A has a bounded right approximate identity, then U is onto.*

(c) *If A has a right approximate identity bounded by one, then for each $T \in M_R(A)$, there exist $m \in J_R(X^*)$ such that $\|m\| = \|T\|$ and $U(m) = T$. Furthermore, if X is a C^*-subalgebra of A^* and T is positive, then m can be chosen to be positive.*

(d) *U is one-one if and only if $\langle X \cdot A \rangle^- = X$.*

(e) *If U is one-one and $J_R(X^*) = X^*$, then $X \subseteq WAP(A) \cap UC_r(A)$.*

Proof. (a) is trivial.

(b) Let $\{\phi_\alpha\}$ be a bounded right approximate identity of A. Then for each $T \in M(A)$, let m be a weak*-cluster point of the net $\{T(\phi_\alpha)\}$ in X^*. Then for each $\phi \in A$, $x \in X$, we have

$$\begin{aligned}\langle T(\phi), x \rangle &= \lim_\alpha \langle T(\phi \cdot \phi_\alpha), x \rangle = \lim_\alpha \langle \phi \cdot T(\phi_\alpha), x \rangle = \lim_\alpha \langle T(\phi_\alpha), x \cdot \phi \rangle \\ &= \langle m, x \cdot \phi \rangle = \langle \phi \odot m, x \rangle = \langle U(m)(\phi), x \rangle.\end{aligned}$$

So $T = U(m)$.

(c) In this case we may choose (by Lemma 4.1) a bounded right approximate identity $\{\phi_\alpha\}$ in A consisting of states. Then if m is a weak*-cluster point of $\{T(\phi_\alpha)\}$, we have $\|m\| \leqslant \|T(\phi_\alpha)\| \leqslant \|T\|$. Consequently, $\|m\| = \|T\|$. Also if X is a C^*-subalgebra of A^* and T is positive, then $T(\phi_\alpha)$ are also positive functionals on X. Hence m is positive.

(d) If $\langle X \cdot A \rangle^- = X$ and $U(m_1) = U(m_2)$, $m_1, m_2 \in J_R(X^*)$, then $m_1(x \cdot \phi) = m_2(x \cdot \phi)$ for each $x \in X$, $\phi \in A$. Hence $m_1 = m_2$. Conversely, if $\langle X \cdot A \rangle^-$ is a proper subspace of X, let $m \in X^*$ such that $m(x \cdot \phi) = 0$ for all $x \in X$, $\phi \in A$ and $m \neq 0$. Then $m \in J_R(X^*)$ and $U(m) = 0$. Hence U is not one-one.

(e) It follows from (d) that $X \subseteq UC_r(A)$. To see that $X \subseteq WAP(A)$, it is sufficient to show that Arens product on X^* is separately continuous in the weak*-topology on bounded spheres [21, Theorem 4.1]. Indeed if $\{n_\beta\}$ is a bounded net in X^* converging to n in the weak*-topology, and $m \in X^*$, then clearly $n_\beta \odot m$ converges to $n \odot m$ in the weak*-topology. Also if $x \in X$, $\psi \in A$, then

$$\langle m \odot n_\beta, x \cdot \psi \rangle = \langle (\psi \odot m) \odot n_\beta, x \rangle = \langle n_\beta, x \cdot (\psi \odot m) \rangle$$

which converges to $\langle n, x \cdot (\psi \odot m) \rangle = \langle m \odot n, x \cdot \psi \rangle$ since $\psi \odot m \in A$. Since $Y = \langle X \cdot A \rangle$ is norm dense in X (by (d)), it follows that the weak*-topology and the $\sigma(X^*, Y)$ agree on bounded spheres. Hence $m \odot n_\beta$ converges to $m \odot n$ in the weak*-topology.

McKilligan proved in [18, Theorem 2.1] that if B is an L-algebra with a (weak) approximate identity bounded by one, and Y is a σ-dense C^*-subalgebra of B^* which is topologically invariant and contained in $WAP(B)$, then the algebra of right multipliers of B is isometric and algebra anti-isomorphic to the largest subalgebra in Y^* which contains the image of the natural embedding of B in Y^* as a right ideal. Notice in this case that Y is topologically introverted [19, Theorem 3.1], and $Y \subseteq UC_r(B)$ (Proposition 3.3 (b)). Our next theorem is an improvement and generalisation of McKilligan's result (see also Remark 4.4).

THEOREM 4.3. *Assume that A has a right approximate identity bounded by one. Then the following are equivalent*:

(a) *U is a linear isometry and algebra anti-isomorphism from $J_R(X^*)$ onto $M_R(A)$.*

(b) *$X \subseteq UC_r(A)$.*

In this case, if X is a C^*-subalgebra of A^*, then $U(m)$ is positive if and only if m is positive.

Proof. That (a) implies (b) follows from Proposition 4.2(d). Conversely, if $X \subseteq UC_r(A)$ and $\{\phi_\alpha\}$ is a right approximate identity in A bounded by one, then for each $x \in X$ we may (by Cohen's factorization theorem) write $x = y \cdot \phi$ for some $y \in A^*$, $\phi \in A$. Then

$$\|x - x \cdot \phi_\alpha\| = \|y \cdot \phi - y \cdot \phi \cdot \phi_\alpha\| \leqslant \|y\| \, \|\phi - \phi \cdot \phi_\alpha\| \to 0.$$

Hence $X = \overline{X \cdot A}$. Consequently (a) and the last statement follow from Proposition 4.2.

Remark 4.4. (a) Theorem 4.3 (except for the last statement) remains valid for any complex Banach algebra A which is the predual of some W^*-algebra.

(b) Theorem 4.3 is false without the existence of right approximate identity. In fact, let M be any von Neumann algebra with dimension greater than one and let $A = M_*$ be the F-algebra with multiplication $\phi \cdot \psi = \phi(e)\psi$, $\phi, \psi \in A$. Then A has left identities but no right approximate identity. Also

$UC_r(A) = A^*$. Let $X = UC_r(A)$. Then $J_R(X^*) = A$, and $M_R(A) = \mathscr{B}(A)$, the algebra of bounded linear operators from A to A. Now U is not onto since the identity operator I is in $M_R(A)$ but $U(m) \neq I$ for any $m \in J_R(X^*)$.

COROLLARY 4.5. *Let A be an F-algebra with a right approximate identity bounded by one. Let X be a topologically invariant and topologically left introverted ultraweakly dense C^*-subalgebra of A^* contained in $UC_r(A)$. Then $M_R(A)$ is isometric and order anti-isomorphic to the largest closed subalgebra of X^* containing A as a right ideal.*

5. The dual algebra $UC_r(A^*)$. When G is a locally compact group, Curtis and Figà-Talamanca [3, Theorem 3.3] has identified the space of bounded linear operators T from $L_\infty(G)$ into $L_\infty(G)$ commuting with convolution by $L_1(G)$ with the dual of $UC_r(G)$. The author [14, Theorem 6.2] and independently Carlo Cecchini (private correspondence) have obtained the analogue result for bounded linear operators from $VN(G)$ into $VN(G)$ commuting the action of $A(G)$ when G is an amenable locally compact group. Our next theorem is a generalisation of these two results to operators on the dual algebra of an F-algebra.

THEOREM 5.1. *Let A be an F-algebra with a right approximate identity bounded by one. Let X be a closed, topologically invariant and topologically left introverted subspace of A^*. Let $Y = X \cdot A$. Then Y is a closed topologically invariant and topologically left introverted linear subspace of $UC_r(A)$. Also the mapping Q: $m \to m_L$ is a linear isometry and algebra isomorphism from Y^* onto the algebra of bounded linear operators T from X into X such that $T(x \cdot \phi) = T(x) \cdot \phi$ for all $x \in X$, $\phi \in A$, where $\langle m_L(x), \phi \rangle = \langle m, x \cdot \phi \rangle$, $m \in Y^*$.*

Proof. That Y is a closed linear space follows from Cohen's factorization theorem [12, 32.22]. Also an argument similar to that for Proposition 3.1 shows that Y is topologically invariant and topologically left introverted.

If $m \in Y^*$, then $m_L(x \cdot \phi) = m_L(x) \cdot \phi$ for all $x \in X$, $\phi \in A$ and $\|m_L\| \leqslant \|m\|$. To see that equality holds, let $\{\phi_\alpha\}$ be a right approximate identity of A bounded by one. Then for each $z \in X \cdot A$, $\|z \cdot \phi_\alpha - z\| \to 0$. Hence

$$\|m_L(z)\| \geqslant |\langle m_L(z), \phi_\alpha \rangle| = |\langle m, z \cdot \phi_\alpha \rangle|$$

which converges to $|\langle m, z \rangle|$. Hence $\|m_L\| \geqslant \|m\|$.

It is easy to see that if $m, n \in Y^*$, then $(n \odot m)_L = n_L(m_L)$. Finally let T be a bounded linear operator from X to X and $T(x \cdot \phi) = T(x) \cdot \phi$ for all $x \in X$, $\phi \in A$, let m be a weak*-cluster point of the net $\{T^*(\phi_\alpha)\}$ in Y^*. Then if $x \in X$, $\gamma \in A$, we have

$$\langle T(x), \gamma \rangle = \lim_\alpha \langle T(x), \gamma \cdot \phi_\alpha \rangle = \lim_\alpha \langle T(x) \cdot \gamma, \phi_\alpha \rangle$$

$$= \lim_\alpha \langle T(x \cdot \gamma), \phi_\alpha \rangle = \langle m_L(x), \gamma \rangle.$$

Hence $T = m_L$, i.e. Q is onto.

COROLLARY 5.2. *Let A be an F-algebra with a right approximate identity bounded by one. Then $UC_r(A)^*$ is an isometric algebra isomorphic to the algebra of bounded operators T from A^* into A^* such that $T(x\cdot\phi) = T(x)\cdot\phi$ for all $x\in A^*$, and $\phi\in A$.*

Remark 5.3. Theorem 5.1 is not true when A does not have a bounded right approximate identity. Indeed, let A be the F-algebra considered in Remark 4.4(b), and $X = A^*$. Then $Y = UC_r(A) = A^*$ and Q is not onto, since there exists no m in A^{**} such that $Q(m)$ is the identity operator on A^*.

6. Invariant means. Let A be an F-algebra and X be a closed topologically invariant subspace of A^* containing e. Then an element $m\in X^*$ is a *topological left invariant mean* (TLIM) on X if $\|m\| = m(e) = 1$, and $m(x\cdot\phi) = m(x)$ for each $\phi\in P_1(A)$, $x\in X$. Topological right invariant mean (TRIM) is defined similarly.

It is well known that when G is a locally compact group, then $L_\infty(G)$ has a TLIM if and only if $UC(G)$ (bounded uniformly continuous functions on G) has a TLIM (see [11, Theorem 2.2.1]). The following is a generalisation of this result:

PROPOSITION 6.1. *Let A be an F-algebra.*

(a) *A^* has a* TLIM *if and only if $UC_l(A)$ has a* TLIM.

(b) *If A has a left approximate identity, then A^* has a* TLIM *if and only if $UC(A)$ has a* TLIM.

Proof. (a) Clearly the restriction of any TLIM of A^* to $UC_l(A)$ is a TLIM. Conversely, if $m\in UC_l(A)^*$ is a TLIM, let $\phi_0\in P_1(A)$ be fixed. Define $\tilde{m}(x) = m(\phi_0\cdot x)$ for each $x\in A^*$. Then $\tilde{m}$ is a TLIM on A^*.

(b) Again we only need to prove that if $UC(A)$ has a TLIM m, then so does A^*. Let $\phi_0\in P_1(A)$ be fixed. We first show that $m(\phi_0\cdot x\cdot\psi_1) = m(\phi_0\cdot x\cdot\psi_2)$ for any $\psi_1, \psi_2\in P_1(A)$, any $x\in A^*$. Indeed, let $\{\phi_\alpha\}$ be a left approximate identity of A, then for each $x\in A^*$ we have:

$$\begin{aligned} m(\phi_0\cdot(x\cdot\psi_1)) &= \lim_\alpha m(\phi_0\cdot(x\cdot\phi_\alpha\cdot\psi_1)) = \lim_\alpha m(\phi_0\cdot x\cdot\phi_\alpha) \\ &= \lim_\alpha m(\phi_0\cdot(x\cdot\phi_\alpha\cdot\psi_2)) = \lim_\alpha m(\phi_0\cdot x\cdot\psi_2). \end{aligned}$$

Let $\psi_0\in P_1(A)$ be fixed, and define $\tilde{m}(x) = m(\phi_0\cdot x\cdot\psi_0)$. Then if $\psi\in P_1(A)$, we have

$$\tilde{m}(x\cdot\psi) = m(\phi_0\cdot x\cdot(\psi\cdot\psi_0)) = m(\phi_0\cdot x\cdot\psi_0) = \tilde{m}(x)$$

by the above.

Remark 6.2. Proposition 6.1(b) is false when A is not assumed to have a left approximate identity. Indeed, let A be the F-algebra defined in Remark 3.5. Then $UC(A) = \{\lambda e;\ \lambda\in C\}$ has a TLIM, but A^* has a TRIM but not a TLIM.

PROPOSITION 6.3. *Let A be an F-algebra. Let X be a topologically invariant closed subspace of $WAP(A)$ containing e. If X has a* TLIM *m and a* TRIM *n, then $m = n$. In particular, if A is commutative, then $WAP(A)$ has a unique topological invariant mean.*

Proof. Let $\tilde{m}$ be an extension of m to A^* with the same norm. Then m is a state of A^*. Let $\phi_\alpha \in P_1(A)$ be a net converging to m in the weak*-topology (Lemma 2.1). Then for each (fixed) $x \in X$, the net $\{\phi_\alpha \cdot x\}$ converges to $m(x)e$ in the σ-topology of A^*. Since $\overline{K(x)}$ is weakly compact, where $K(x) = \{\phi \cdot x;\ \phi \in P_1(A)\}$, it follows that the weak topology and the σ-topology agree on $\overline{K(x)}$. Consequently $m(x)e \in \overline{K(x)}$. So we may find a net $\{\psi_\beta \cdot x\}$, $\psi_\beta \in P_1(A)$, converging to $m(x)e$ in the norm topology. Similarly, we can find a net $\{x \cdot \eta_\gamma\}$, $\eta_\gamma \in P_1(A)$, converging to $n(x)e$ in norm. Hence

$$\begin{aligned} |m(x) - n(x)| &= \|m(x)e - n(x)e\| \\ &\leqslant \|(m(x)e - \psi_\beta \cdot x) \cdot \eta_\gamma\| + \|\psi_\beta \cdot (x \cdot \eta_\gamma - n(x)e)\| \to 0. \end{aligned}$$

Consequently $m = n$. The last assertion follows from Example after Corollary 4.3 [15].

REFERENCES

[1] R. F. Arens, *The adjoint of a bilinear operation*, Proceedings of the American Mathematical Society 2 (1951), p. 839–848.

[2] P. Civin and B. Yood, *The second conjugate space of Banach algebra as an algebra*, Pacific Journal of Mathematics 11 (1961), p. 847–870.

[3] P. C. Curtis and A. Figà-Talamanca, *Factorization Theorems for Banach algebras*, in: *Function Algebras* (Proc. International Symposium on Function Algebras, Tulane Univ., 1965), Scott–Foreman, Chicago, Ill., 1966, p. 169–185.

[4] N. Dunford and J. T. Schwartz, *Linear Operators I*. Interscience Publishers, 1957.

[5] C. F. Dunkl, *The measure algebra of a locally compact hypergroup*, ibidem 179 (1973), p. 331–348.

[6] – and D. E. Ramirez, *Weakly almost periodic functionals on the Fourier Algebra*, Transactions of the American Mathematical Society 185 (1973), p. 501–514.

[7] E. G. Effros, *Order ideals in a C^*-algebra and its dual*, Duke Mathematical Journal 30 (1963), p. 391–411.

[8] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bulletin de la Société Mathématique de France 92 (1964), p. 181–236.

[9] E. Granirer, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*, Transactions of the American Mathematical Society 189 (1974), p. 371–382.

[10] –, *Density theorems for some linear subspaces and some C^*-subalgebras of $VN(G)$*. Symposia Mathematica (1977) Vol. XXII, Istituto Nazionale di Alta Matematica, p. 61–70.

[11] F. P. Greenleaf, *Invariant Means on Topological Groups and Their Applications*. Van Nostrand Reinhold, New York 1969.

[12] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, 1963.

[13] –, –, *Abstract Harmonic Analysis II*, Springer-Verlag 1970.

[14] A. T. Lau, *Uniformly continuous functionals on the Fourier algebra of any locally compact group*, Transactions of the American Mathematical Society 251 (1979), p. 39–59.
[15] —, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups*, Fundamenta Mathematicae 118 (1983), p. 161–175.
[16] H. Leptin, *Sur l'algèbre de Fourier d'un localement compact*, Comptes Rendus des Séances de l'Académie des Sciences. Série A–B 266 (1968), p. 1180–1182.
[17] K. McKennon, *Multipliers, positive functionals, positive-definite functions, and Fourier–Stieltjes transforms*, Memoirs of the American Mathematical Society 111 (1971).
[18] S. A. McKilligan, *On the representation of multiplier algebras of some Banach algebras*, The Journal of the London Mathematical Society (2) 6 (1973), p. 399–402.
[19] — and A. J. White, *Representations of L-algebras*, Proceedings of the London Mathematical Society 25 (1972), p. 655–674.
[20] I. Namioka, *Følner's conditions for amenable semi-groups*, Mathematica Scandinavica 15 (1964), p. 18–28.
[21] J. S. Pym, *The convolution of functionals on spaces of bounded functions*, Proceedings of the London Mathematical Society 15 (1965), p. 84–104.
[22] J. L. Taylor, *The structure of convolution measure algebras*, Transactions of the American Mathematical Society 119 (1965), p. 150–166.
[23] J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific Journal of Mathematics 2 (1952), p. 251–261.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA,
CANADA

Reçu par la Rédaction le 03. 02. 1984