COLLOQUIUM MATHEMATICUM

VOL. LX/LXI DÉDIÉ À M. ANTONI ZYGMUND 1990

CALDERÓN-ZYGMUND OPERATORS AND POISSON-LIKE OPERATORS

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1. Introduction. In a number of papers (cf. [4], [6], [7], [12], [13]), a theory of tent spaces has been developed, with interesting applications to harmonic analysis. The reader is referred to [5], [16] for a survey on tent spaces and also to [1], [23], [24] for more applications.

In this paper, we study a class of operators related to the Poisson transform, in the context of tent spaces and H^p spaces. Following the notation in [21], p. 308, we call these operators Poisson-like operators. They are closely related to the Calderón–Zygmund theory. Indeed, the Littlewood–Payley decomposition of an operator associated with a standard kernel (cf. [10]) is, under certain conditions, a family of Poisson-like operators. One could roughly say that to prove the celebrated T(1) Theorem (cf. [9], [14]) is to prove a suitable continuity result for a Poisson-like operator whose kernel has some cancellation. In the same spirit, paraproducts (cf. [8]) are also closely related to Poisson-like operators. Furthermore, the notion of Poisson-like operator plays a crucial role in [9].

The connection between tent spaces and H^p spaces is provided by a convolution-like operator, π_φ, introduced in [13]. This operator is a particular case of a Poisson-like operator. However, for the end point spaces T^p_∞, $0 < p \leq 1$, this operator is not continuous from T^p_∞ into H^p. Nevertheless, since $T^{p,q}_\infty$ is continuously included in $L^q(\mathbf{R}^{n+1}_+, \mu)$, where μ is a Carleson measure of order q/p, a natural possible substitute for these operators can be obtained by replacing the measure $dxdt/t$ by a positive Borel measure $\mu(y,t)$ defined on $\mathbf{R}^{n+1}_+$. Then, the corresponding Poisson-like operator acting on a function $f(y,t)$ is the composition of two operators: multiplication of the measure $\mu(y,t)$ by the function $f(y,t)$, which produces a new measure $fd\mu$, and then a balayage of this measure (cf. [6]). For example, if $f \in T^p_\infty$, $0 < p \leq 1$ and $\mu \in V^{1/p}$ (cf. [15]), by the duality theory (cf. [4], [7]), $fd\mu$ is a finite measure. The balayage of a finite measure is in L^1.

From this point of view, Poisson-like operators extend the ones considered in [6]. In fact, the methods developed in [6] play an essential role here.

Another important connection is given by [19], where essentially the adjoints of our operators are studied in the context of weighted L^p spaces. We give a different approach to some of the results in [13] and we are able to consider a different set of indexes as well.

Our paper is divided into three sections. In Section 2, we make precise the definitions involving Poisson-like operators and we describe in detail the connections with the Calderón–Zygmund theory and the operator π_φ. In Section 3, we prove continuity results in tent spaces, H^p spaces and Lipschitz spaces. Sometimes our techniques will resemble the methods used in the Calderón–Zygmund theory, while other results will be proved in the spirit of balayages.

The notation used in this paper is standard. It may be useful to point out that given $1 < p < \infty$, p' will denote the conjugate exponent, $1/p + 1/p' = 1$. Also, when there is no further explanation, the usual symbols C_0^∞, $\mathcal{D}'$, L^p, etc. will refer to spaces of functions defined on $\mathbf{R}^n$, with respect to the Lebesgue measure. Otherwise, the space will be denoted $L^p(\mathbf{R}_+^{n+1}, \mu)$, $L^\infty(\mathbf{R}_+^{n+1})$, etc. The subindex 0 will indicate a space of functions with compact support.

2. Poisson-like kernels and Poisson-like operators

DEFINITION (2.1) (cf. [6], p. 31). Let $k(x,y,t)$ be a continuous function, $k : \mathbf{R}^n \times \mathbf{R}_+^{n+1} \to \mathbf{C}$. The function is a *Poisson-like kernel* if it satisfies the following conditions, for some $0 < \delta \le 1$:

$$(\mathrm{C}_\delta) \qquad |k(x,y,t)| \le ct^\delta/(|x-y|+t)^{n+\delta},$$

$$(\mathrm{D}_\delta) \qquad |k(x,y,t) - k(z,y,t)| \le c|x-z|^\delta/(|y-z|+t)^{n+\delta}$$

$$\text{if } 2|x-z| < |y-z| + t.$$

EXAMPLE (2.2). Let $\varphi : \mathbf{R}^n \to \mathbf{C}$ be a C^1 function. Suppose that

$$|\varphi(x)| + |\nabla\varphi(x)| \le c(1+|x|)^{-n-1} \tag{2.3}$$

for some $c > 0$. Then $\varphi_t(x-y) = t^{-n}\varphi((x-y)/t)$ is a Poisson-like kernel, for $\delta = 1$.

Proof. Let us first prove condition (C_1). According to (2.3),

$$|\varphi_t(x-y)| \le ct^{-n}(1+|x-y|/t)^{-n-1} = ct(1+|x-y|)^{-n-1}.$$

Let us now prove condition (D_1). We have

$$|\varphi_t(x-y)-\varphi_t(z-y)| = \frac{1}{t^{n+1}}\left|\int_0^1 (\nabla\varphi)\Big(\frac{z-y}{t}+s\frac{x-z}{t}\Big)\cdot(x-z)\,ds\right|$$

$$\le c\frac{|x-z|}{t^{n+1}}\int_0^1 \frac{ds}{\Big(\frac{|z-y|}{t}+s\frac{|x-z|}{t}+1\Big)^{n+1}}.$$

If $2|x-z|<|y-z|+t$, then $|z-y+s(x-z)|+t\ge |y-z|-|x-z|+t\ge \frac{1}{2}(|y-z|+t)$. Thus,

$$|\varphi_t(x-y)-\varphi_t(z-y)|\le c|x-z|/(|y-z|+t)^{n+1}.$$

This completes the proof.

It should be noted that the Poisson kernel itself is a particular case of Example (2.2). Another important example is the kernel of the operator π_φ, introduced by R. Coifman, Y. Meyer and E. Stein (cf. [13]) to provide the link between the tent space T_2^p and the Hardy space H^p when $0<p<\infty$ or the space BMO when $p=\infty$. For future reference, we will now state precisely the definition of π_φ.

Consider a C^1 function satisfying (2.3). Suppose that φ also satisfies a cancellation condition, namely,

(2.4) $$\int x^\gamma\varphi(x)\,dx = 0, \qquad 0\le|\gamma|\le N,$$

for some $N=0,1,\ldots$ Now, set as usual, $\varphi_t(x)=t^{-n}\varphi(x/t)$ and define, at least formally,

(2.5) $$\pi_\varphi(f) = \int_0^\infty f(\ ,t)*\varphi_t\,\frac{dt}{t}$$

given $f:\mathbf{R}_+^{n+1}\to\mathbf{C}$, or

(2.6) $$\pi_\varphi(f)(x) = \int_{\mathbf{R}_+^{n+1}} \varphi_t(x-y)f(y,t)\frac{dy\,dt}{t}$$

where $\varphi_t(x-y)$ is a Poisson-like kernel.

EXAMPLE (2.7). This example deals with paraproducts (cf. [10], Appendix I). It is not difficult to show that given a function $b\in$ BMO, the operator M_b of pointwise multiplication by b will not, in general, be continuous on L^2. A paraproduct is essentially a redefinition of the operator M_b, to give a bilinear, continuous action from $L^2\times$ BMO into L^2.

Paraproducts are related with a very interesting subclass of the Hörmander class $L_{1,1}^m$, called paradifferential operators (cf. [8], [15], [20]).

Paraproducts also play a crucial role in the original proof of the T(1) Theorem (cf. [14]). Indeed, they are used to reduce the given operator to one satisfying vanishing conditions, $T(1) = T^*(1) = 0$. More precisely, it suffices to show that given $b \in$ BMO, there exists a Calderón–Zygmund operator L such that $L(1) = b$, $L^*(1) = 0$. There are many ways of constructing L. But whatever definition is chosen, the notion of Poisson-like operator appears as a central one in the estimates.

As an example, we will now give a definition of L as a paraproduct due to E. Stein. Let φ, $\psi \in \mathcal{S}$ be radial functions satisfying

$$\begin{aligned} &\operatorname{supp}(\hat{\varphi}) \subset \{|\xi| \le 1/4\}, \qquad \int \varphi\, dx = 1, \\ &\operatorname{supp}(\hat{\psi}) \subset \{1 \le |\xi| \le 2\}, \quad \int \hat{\psi}(t\xi)\,\frac{dt}{t} = 1, \quad \xi \neq 0. \end{aligned} \tag{2.8}$$

Given $f \in C_0^\infty$, consider, formally,

$$L(f)(x) = \int_0^\infty (\varphi_t * f)(x)\,(\psi_t * b)(x)\,\frac{dt}{t}. \tag{2.9}$$

It can be proved (cf. [2], p. 20 for details) that (2.9) defines, in the weak sense, a Calderón–Zygmund operator satisfying

$$\|L(f)\|_{L^2} \le c\|b\|_{\mathrm{BMO}}\|f\|_{L^2}, \qquad L(1) = b,\ L^*(1) = 0. \tag{2.10}$$

It is central to the proof of (2.10) to identify the kernel of (2.9) and to estimate it.

We can write, formally,

$$L(f)(x) = \int_0^\infty \Big(\int \varphi_t(x-y)f(y)\,dy\Big)(\psi_t * b)(x)\frac{dt}{t},$$

or

$$L(f)(x) = \int_{\mathbf{R}^{n+1}_+} \varphi_t(x-y)(\psi_t * b)(x)f(y)\,\frac{dy\,dt}{t}. \tag{2.11}$$

We claim that the kernel

$$L(x,y,t) = \varphi_t(x-y)(\psi_t * b)(x) \tag{2.12}$$

is a Poisson-like kernel, for $\delta = 1$. More precisely, we will prove the following estimates:

$$|L(x,y,t)| \le ct/(|x-y|+t)^{n+1}, \tag{2.13}$$

$$|\nabla_{x,y} L(x,y,t)| \le c/(|x-y|+t)^{n+1}. \tag{2.14}$$

Condition (D_1) will easily follow from (2.14).

Proof. Since $\varphi \in \mathcal{S}$, we have $|\varphi(x-y)| \leq c/(1+|x-y|)^{n+1}$, or $|\varphi_t(x-y)| \leq ct/(|x-y|+t)^{n+1}$. On the other hand,

$$|(\psi_t * b)(x)| = \left|\frac{1}{t^n} \int_{|x-z|\leq 2t} \psi_t(x-z)b(z)\,dz\right|.$$

If $b(x,t)$ denotes the average of the function b over the ball $B(x,2t)$, of center x and radius $2t$, the integral above can be majorized by

$$c\,\|\psi\|_{L^\infty} \frac{1}{|B(x,2t)|} \int_{B(x,2t)} |b(z) - b(x,t)|\,dz \leq c\,\|\psi\|_{L^\infty}\,\|b\|_{\mathrm{BMO}}\,.$$

This proves (2.13) when ψ has compact support. Now

$$\nabla_x L(x,y,t) = \frac{1}{t}(\nabla\varphi)_t(x-y)(\psi_t * b)(x) + \varphi_t(x-y)\frac{1}{t}((\nabla\psi)_t * b)(x),$$

$$\nabla_y L(x,y,t) = -\frac{1}{t}(\nabla\varphi)_t(x-y)(\psi_t * b)(x).$$

Thus, the same computations as before show that (2.14) also holds. This completes the proof of the claim.

EXAMPLE (2.15). The previous example dealt with the original proof of the T(1) Theorem. We will show how the notion of Poisson-like kernel also appears in the shorter proof due to R. Coifman and Y. Meyer (cf. [11]).

In fact, let $T : C_0^\infty \to \mathcal{D}'$ be a linear and continuous operator and suppose that the distribution kernel $k(x,y)$ defined by T is a continuous function on $\mathbf{R}^n \times \mathbf{R}^n \backslash$diagonal, satisfying the standard estimates (cf. [10], p. 78)

$$|k(x,y)| \leq c/|x-y|^n, \tag{2.16}$$

$$|k(x,y) - k(z,y)| + |k(y,x) - k(y,z)| \leq c\frac{|x-z|^\varepsilon}{|y-z|^{n+\varepsilon}} \tag{2.17}$$

$$\text{if } 2|x-z| \leq |y-z|, \text{ for some } 0 < \varepsilon \leq 1.$$

Suppose also that T satisfies the *weak boundedness property* (WBP): Given $\mathbf{B} \subset C_0^\infty$ a bounded subset, there is $c > 0$ such that for any φ, $\psi \in \mathbf{B}$, $z \in \mathbf{R}^n$, $t > 0$, we have,

$$\left|\left(T\varphi\Big(\frac{y-z}{t}\Big), \psi\Big(\frac{x-z}{t}\Big)\right)\right| \leq ct^n. \tag{2.18}$$

Then, under the above conditions, the following representation formula for T holds (cf. [11]).

Let $\varphi \in C_0^\infty$ be a radial function such that $\operatorname{supp}(\varphi) \subset \{|x| \leq 2\}$, $\varphi(x) = 1$ if $|x| \leq 1$, $0 \leq \varphi \leq 1$, $\int \varphi\,dx = 1$. Let $P_t(f) = \varphi_t * f$. Notice that according to Example (2.2), φ_t is a Poisson-like kernel. But, also, φ_t is somehow a nicer kernel, since φ has compact support. Nevertheless, not all the conditions imposed on φ are essential.

Now, in the weak sense, we can write

$$T = -\int_0^\infty \frac{d}{dt}(P_t T P_t)\,\frac{dt}{t} \tag{2.19}$$

(cf. [2], p. 29, for details). Let us examine closer the integral in (2.19). In the weak sense,

$$\frac{d}{dt}(P_t T P_t) = \Big(\frac{d}{dt}\varphi_t * \ \Big) T P_t + P_t T \Big(\frac{d}{dt}\varphi_t * \ \Big).$$

But

$$\frac{d}{dt}\varphi_t(x) = \frac{d}{dt}\Big(\frac{1}{t^n}\varphi\Big(\frac{x}{t}\Big)\Big) = -\frac{1}{t}\Big[\sum_j \frac{\partial}{\partial x_j}(x_j\varphi)\Big]_t = -\frac{1}{t}\psi_t(x),$$

where $\int \psi\, dx = 0$. Let $Q_t = \psi_t *$. Thus, in the weak sense, the following Littlewood–Paley decomposition holds:

$$T = \int_0^\infty (Q_t T P_t + P_t T Q_t)\,\frac{d}{dt}. \tag{2.20}$$

We will now show that the two operators $Q_t T P_t$, $P_t T Q_t$ are defined by almost Poisson-like kernels. Indeed, since $(P_t T Q_t)^* = Q_t T^* P_t$ and T, T^* satisfy the same conditions, it suffices to consider $Q_t T P_t$. For $t > 0$ fixed, $Q_t T P_t$ is certainly a linear and continuous operator from C_0^∞ into $\mathcal{D}'$. Moreover, it is an integral operator, formally

$$Q_t T P_t(f)(x) = \int_{\mathbf{R}_+^{n+1}} k(x,y,t) f(y,t)\,\frac{dy\,dt}{t} \tag{2.21}$$

where the kernel $k(x,y,t)$ is the $C^\infty(\mathbf{R}^n \times \mathbf{R}_+^{n+1})$ function given by

$$k(x,y,t) = (T\varphi_t(v-y), \psi_t(u-x)). \tag{2.22}$$

We claim that $k(x,y,t)$ satisfies condition (C_ε) as well as the *almost* (D_ε) *condition*:

$$(\mathrm{AD}_\varepsilon) \qquad |k(x,y,t) - k(z,y,t)| \le c\frac{|x-z|}{t}\,\frac{t^\varepsilon}{(|y-z|+t)^{n+\varepsilon}} \quad \text{if } 2|x-z| < |y-z| + t.$$

Observe that condition $(\mathrm{AD}_\varepsilon)$ coincides with condition (D_ε) when $|x-z| < t$ or when $\varepsilon = 1$.

Let us first prove condition (C_ε). We will consider two cases. First, suppose that $|x-y| \le at$, for some $a > 0$ to be fixed. We observe that $\psi_t(u-x) = t^{-n}\psi(W + (u-y)/t)$, where $W = (y-x)/t$. Therefore,

$\{\varphi, \psi(W+\ )\}_{|W|\leq a}$ is a bounded subset of C_0^∞. Since T has the WBP, from (2.22) we get

$$|k(x,y,t)| \leq c/t^n \leq ct^\varepsilon/(|x-y|+t)^{n+\varepsilon}.$$

Now, suppose that $|x-y| > at$. Since $\operatorname{supp}\varphi_t(u-y) \subset \{|u-y| \leq 2t\}$, $\operatorname{supp}\psi_t(u-x) \subset \{|u-x| \leq 2t\}$, they will be disjoint if we select, say, $a = 6$. Thus, we can write

$$k(x,y,t) = \int k(u,v)\varphi_t(v-y)\psi_t(u-x)\,du\,dv. \tag{2.23}$$

Since $\int \psi\,dx = 0$, we can also write

$$k(x,y,t) = \int (k(u,v)-k(x,v))\varphi_t(v-y)\psi_t(u-x)\,du\,dv.$$

With all our hypothesis, $|x-v| \geq |x-y| - |y-v| \geq 4t \geq 2|x-u|$ and also, $|x-v| \geq |x-y| - 2t \geq \frac{2}{3}|x-y|$. Thus, $|k(u,v)-k(x,v)| \leq c|x-u|^\varepsilon/|x-y|^{n+\varepsilon}$ and then

$$|k(x,y,t)| \leq c\frac{t^\varepsilon}{|x-y|^{n+\varepsilon}} \leq c\frac{t^\varepsilon}{(|x-y|+t)^{n+\varepsilon}}.$$

Let us now prove $(\mathrm{AD}_\varepsilon)$. We have

$$\begin{aligned} k(x,y,t) - k(z,y,t) &= \int_0^1 (\nabla_x k)(z+s(x-z),y,t)\cdot(x-z)\,ds \\ &= \int_0^1 (T\varphi_t(v-y), t^{-1}(\nabla_x\psi)_t(u-z-s(x-z))\cdot(x-z))\,ds. \end{aligned}$$

We use condition (C_ε) with the kernel defined by

$$(T\varphi_t(v-y), t^{-1}(\nabla_x\psi)_t(u-z-s(x-z))\cdot(x-z))$$

to obtain

$$|k(x,y,t)-k(z,y,t)| \leq c\frac{|x-z|}{t}\int_0^1 \frac{t^\varepsilon}{(t+|z+s(x-z)-y|)^{n+\varepsilon}}\,ds.$$

But since we are supposing that $2|x-z| < |y-z|+t$, we have $t + |z + s(x-z) - y| \geq t + |y-z| - |x-z| \geq \frac{1}{2}(t+|y-z|)$. Thus, $k(x,y,t)$ satisfies condition $(\mathrm{AD}_\varepsilon)$. This completes the proof.

Remarks (2.24). (i) The formula (2.21) will rarely be well defined, unless some cancellation is built up into the kernel. This can be accomplished by either reducing the operator T to vanishing conditions, when $T(1)$, $T^*(1) \in$ BMO (cf. [14]), or by adopting a more refined version of the representation formula (2.20) (cf. [11]). The problem is caused by the presence of the measure $dy\,dt/t$, which is not locally finite near $t = 0$.

Observe that the operator π_φ defined in Example (2.2) has this cancellation property, due to condition (2.4).

(ii) With the same proof, it is possible to show that the function $k(x, y, t)$ given by (2.22) also satisfies

$$(2.25)\qquad |k(y,x,t) - k(y,z,t)| \le c\frac{|x-z|}{t}\,\frac{t^\varepsilon}{(|y-z|+t)^{n+\varepsilon}} \quad \text{if } 2|x-z| \le |y-z| + t.$$

(iii) It is interesting to point out that given an almost Poisson-like kernel $k(x, y, t)$ satisfying (2.25), the integral

$$(2.26)\qquad \int_0^\infty k(x,y,t)\,\frac{dt}{t}$$

will converge to a standard kernel $k(x, y)$ satisfying (2.16) and (2.17) for $0 < \varepsilon < \delta$ (cf. [17], p. 26).

Proof. Let us first prove (2.16). Using condition (C_δ), we have

$$\begin{aligned}\int_0^\infty |k(x,y,t)|\,\frac{dt}{t} &\le c\int_0^\infty \frac{t^\delta}{(|x-y|+t)^{n+\delta}}\,\frac{dt}{t}\\ &= \frac{c}{|x-y|^n}\int_0^\infty \frac{s^\delta}{(1+s)^{n+\delta}}\,\frac{ds}{s}\\ &= \frac{c}{|x-y|^n}.\end{aligned}$$

Let us now prove (2.17). We only consider the first part, the proof of the second part being the same.

Suppose $2|x-z| < |y-z|$. Then

$$\begin{aligned}|k(x,y) - k(z,y)| \le &\int_0^{|x-z|} |k(x,y,t) - k(z,y,t)|\,\frac{dt}{t}\\ &+ \int_{|x-z|}^\infty |k(x,y,t) - k(z,y,t)|\,\frac{dt}{t}\\ =&I_1 + I_2,\end{aligned}$$

where

$$I_1 \le \int_0^{|x-z|} |k(x,y,t)|\,\frac{dt}{t} + c\int_0^{|x-z|} |k(z,y,t)|\,\frac{dt}{t}$$

$$\leq c \int_0^{|x-z|} \frac{t^\delta}{(|x-y|+t)^{n+\delta}} \frac{dt}{t} + c \int_0^{|x-z|} \frac{t^\delta}{(|y-z|+t)^{n+\delta}} \frac{dt}{t}$$

$$\leq c \int_0^{|x-z|} \frac{t^\delta}{(|y-z|+t)^{n+\delta}} \frac{dt}{t}$$

$$\leq \frac{c}{|y-z|^n} \int_0^{|x-z|/|y-z|} \frac{s^\delta}{(1+s)^{n+\delta}} \frac{ds}{s} \leq c \frac{|x-z|^\delta}{|y-z|^{n+\delta}},$$

$$I_2 \leq c \int_{|x-z|}^{\infty} \frac{|x-z|}{t} \frac{t^\delta}{(|y-z|+t)^{n+\delta}} \frac{dt}{t}$$

$$= c \frac{|x-z|}{|y-z|^{n+1}} \int_{|x-z|/|y-z|}^{\infty} \frac{s^\delta}{(1+s)^{n+\delta}} \frac{ds}{s^2} \leq c \frac{|x-z|^\varepsilon}{|y-z|^{n+\varepsilon}}$$

for any $0 < \varepsilon < \delta$, with $c = c(\varepsilon) > 0$. This completes the proof.

Let us point out that conditions (C_δ), (D_δ) and (2.17) become stronger as $\delta, \varepsilon \to 1$.

(iv) Conditions (C_δ) and (D_δ) can sometimes be replaced by integral estimates in the spirit of [18] (cf. [19], p. 215).

Formulas (2.6), (2.11), (2.21), as well as the notion of balayage introduced in [6], suggest the following definition.

DEFINITION (2.27). Let $T : L_0^\infty(\mathbf{R}_+^{n+1}) \to \mathcal{D}'$ be a linear and continuous operator. We say that T is a *Poisson-like operator* if there exists a Poisson-like kernel $k(x, y, t)$ and a nonnegative Borel measure $\mu(y, t)$ defined on $\mathbf{R}_+^{n+1}$ such that

$$T(f)(x) = T_{k,\mu}(f)(x) = \int_{\mathbf{R}_+^{n+1}} k(x, y, t) f(y, t)\, d\mu(y, t) \tag{2.28}$$

a.e. in $\mathbf{R}^n$.

Poisson-like operators can be viewed as formal adjoints of operators considered in [19]. Indeed, given a Poisson-like kernel $k(x, y, t)$ we can define (cf. [19])

$$K_t(g)(y) = \int k(x, y, t) g(x)\, dx, \qquad g \in L^\infty. \tag{2.29}$$

3. Continuity properties. As mentioned in the introduction, our purpose is to study the continuity of Poisson-like operators defined by generalized Carleson measures, in the context of tent spaces, H^p spaces and Lipschitz spaces. Some of these results can be obtained by simply reading off known results on balayages, combined with some interpolation results (cf. [4], [6], [17], [13]). Sometimes, it will suffice to suppose that the kernel $k(x,y,t)$ satisfies condition (C_δ).

For example, if $f \in T^p_\infty$, $1 < p < \infty$ and $\mu \in V^{1/p}$, then the duality theory implies that $f d\mu$ is a finite measure. Thus, we obtain the result

$$\|T_{k,\mu}(f)\|_{L^1} \le c\, \|\mu\|_{V^{1/p}}\, \|f\|_{T^p_\infty} \tag{3.1}$$

provided that $k(x,y,t)$ satisfies condition (C_δ).

When μ is a Carleson measure and $k(x,y,t)$ meets the (C_δ) condition, the operator $T_{k,\mu}$ satisfies

$$\|T_{k,\mu}(f)\|_{L^p} \le c\, \|\mu\|_{V^1}\, \|f\|_{L^p(\mathbf{R}^{n+1}_+,\mu)}. \tag{3.2}$$

This follows by duality from the very definition of a Carleson measure in terms of the continuity of the Poisson transform.

The space T^p_∞ usually plays an auxiliary role in the study of Poisson-like operators. Let us recall the mechanism involved.

Let $f(y,t)$ be a continuous function on $\mathbf{R}^{n+1}_+$ and let μ be a Carleson measure of order $\alpha > 0$. Then

$$\mu\{(y,t) \mid |f(y,t)| > \lambda\} \le c|\{x \mid A_\infty(f)(x) > \lambda\}|^\alpha \tag{3.3}$$

where A_∞ denotes the nontangential maximal function. Thus,

$$\|f\|_{L^{p\alpha}(\mathbf{R}^{n+1}_+,\mu)} \le c\, \|\mu\|_{V^\alpha}\, \|A_\infty(f)\|_{L^{p,p\alpha}}. \tag{3.4}$$

Particularly, we can carry the above analysis when $f(y,t) = K_t(g)(y)$, given a kernel $k(x,y,t)$ satisfying (C_δ). Since the (C_δ) condition implies that

$$A_\infty(K_t(g))(x) \le cM(g)(x) \tag{3.5}$$

where M is the Hardy–Littlewood maximal operator, we can combine all the above with the maximal theorem to conclude the following.

THEOREM (3.6). *Let $k(x,y,t)$ be a kernel satisfying (C_δ) and let μ be a Carleson measure of order $\alpha > 0$. Then, given $1 < p < \infty$, the following continuity properties hold:*

(i) $K_t : L^{p,p\alpha} \to L^{p\alpha}(\mathbf{R}^{n+1}_+,\mu)$.

(ii) $K_t : L^p \to T^p_\infty$.

(iii) $T_{k,\mu} : L^p(\mathbf{R}^{n+1}_+,\mu) \to L^q$, *provided that $\alpha \ge 1$, $\alpha q' = p'$. Thus, $\alpha < p' < \infty$.*

When $\alpha \ge 1$, part (i) implies that K_t maps L^p into $L^{p\alpha}(\mathbf{R}^{n+1}_+,\mu)$. This was proved in [13], with a different approach. Part (iii) is the dual version of this result.

We will now define two maximal functions. One generalizes the sharp maximal function of C. Fefferman and E. Stein (cf. [22], p. 199) and the other generalizes the maximal function C_q (cf. [13]).

DEFINITION (3.7). (i) Let $\lambda \geq 0$, $1 \leq q < \infty$, $f \in L^q_{\mathrm{loc}}$. Then

$$S_{\lambda,q}(f)(x) = \sup_B \inf_{c\in\mathbf{C}} \Big(\frac{1}{|B|^\lambda} \int_B |f(y) - c|^q\, dy\Big)^{1/q} \tag{3.8}$$

where the supremum is taken over all balls containing x.

(ii) Let $\mu(y,t)$ be a nonnegative Borel measure on $\mathbf{R}^{n+1}_+$ and let $\alpha > 0$, $f \in L^\infty_0(\mathbf{R}^{n+1}_+)$. Then

$$C_{\alpha,q}(f,\mu)(x) = \sup_B \Big(\frac{1}{|B|^\alpha} \int_{T(B)} |f(y,t)|^q\, d\mu(y,t)\Big)^{1/q} \tag{3.9}$$

where the supremum is taken over all balls containing x and $T(B)$ denotes the tent over B.

Remarks (3.10). (i) The condition $S_{\lambda,q}(f) \in L^\infty$ defines the space $L_{\lambda,q}$. The following characterizations are known (cf. [22], p. 213):

$$\begin{aligned} L_{\lambda,q} &= L^q, & \lambda &= 0,\\ L_{\lambda,q} &= \mathrm{BMO}, & \lambda &= 1,\\ L_{\lambda,q} &= \mathrm{Lip}_{n(\lambda-1)/q}, & 1 &< \lambda < 1 + q/n. \end{aligned}$$

(ii) The condition $C_{\alpha,1}(1,\mu) \in L^\infty$ describes the space V^α of generalized Carleson measures, at least when $\alpha \geq 1$.

The next result shows the connection between the maximal functions $S_{\lambda,q}(T_{k,\mu}(f))$ and $C_{\alpha,q}(f,\mu)$. It resembles the pointwise estimate of $(T(f))^{\#}$ in terms of the Hardy–Littlewood maximal operator, when T is a Calderón–Zygmund operator (cf. [10]).

THEOREM (3.11). *Let $T_{k,\mu}$ be a Poisson-like operator and suppose that it maps continuously $L^p(\mathbf{R}^{n+1}_+,\mu)$ into L^q for some $1 \leq p,q < \infty$. Then, given $\lambda \in \mathbf{R}$, $f \in L^\infty_0(\mathbf{R}^{n+1}_+)$,*

$$S_{\lambda,q}(T_{k,\mu}(f))(x) \leq cC_{\lambda p/q,p}(f,\mu)(x) + cC_{\lambda/q-1/q+1,1}(f,\mu)(x) \tag{3.12}$$

provided that $\delta + (n/q)(1-\lambda) > 0$.

Proof. Let $B = B(z,\sigma)$ be a ball, $x \in B$. Given $f \in L^\infty_0(\mathbf{R}^{n+1}_+)$, we consider the usual decomposition

$$f = f\chi_{T(2B)} + f(1 - \chi_{T(2B)}) = f_1 + f_2$$

where $2B$ denotes the ball $B(z,2\sigma)$. Let us also choose

$$c = \int_{\mathbf{R}_+^{n+1}\setminus T(2B)} k(z,y,t)f(y,t)\,d\mu(y,t).$$

Observe that according to the hypothesis, c is well defined. Thus,

$$\Big(\frac{1}{|B|^\lambda}\int_B |T_{k,\mu}(f)-c|^q\,du\Big)^{1/q}$$
$$\le \frac{1}{|B|^{\lambda/q}}\Big(\int_B |T_{k,\mu}(f_1)|^q\,du\Big)^{1/q} + \frac{1}{|B|^{\lambda/q}}\Big(\int_B |T_{k,\mu}(f_2)-c|^q\,du\Big)^{1/q}.$$

The first term can be majorized by

$$\frac{c}{|B|^{\lambda/q}}\Big(\int_{T(2B)} |f(y,t)|^p\,d\mu(y,t)\Big)^{1/p}$$
$$\le c\Big(\frac{1}{|B|^{\lambda p/q}}\int_{T(2B)} |f(y,t)|^p\,d\mu(y,t)\Big)^{1/p} \le cC_{\lambda p/q,p}(f,\mu)(x),$$

which is the first term in (3.12). Consider now the second term:

$$\frac{1}{|B|^{\lambda/q}}\Big(\int_B |T_{k,\mu}(f_2)-c|^q\,du\Big)^{1/q}$$
$$\le \frac{1}{|B|^{\lambda/q}}\Big(\int_B\Big(\int_{\mathbf{R}_+^{n+1}\setminus T(2B)} |k(u,y,t)-k(z,y,t)|\,|f(y,t)|\,d\mu\Big)^q du\Big)^{1/q}.$$

If $(y,t)\notin T(2B)$, by definition of $T(2B)$, there exists $v\in B(y,t)\setminus B(z,2\sigma)$; that is to say, $|y-v|<t$, $|v-z|\ge 2\sigma$, or $|y-z|+t>2\sigma>2|u-z|$. Then we use condition (D_δ) to obtain

$$\frac{1}{|B|^{\lambda/q}}\Big(\int_B\Big(\int_{\mathbf{R}_+^{n+1}\setminus T(2B)} \frac{|u-z|^\delta}{(|y-z|+t)^{n+\delta}}|f(y,t)|\,d\mu\Big)^q du\Big)^{1/q}$$
$$\le \frac{1}{|B|^{\lambda/q-\delta/n-1/q}}\int_{\mathbf{R}_+^{n+1}\setminus T(2B)} \frac{|f(y,t)|}{(|y-z|+t)^{n+\delta}}\,d\mu(y,t)$$
$$= \frac{1}{|B|^{\lambda/q-\delta/n-1/q}}\sum_{j\ge1}\int_{T(2^{j+1}B)\setminus T(2^jB)} \frac{|f(y,t)|}{(|y-z|+t)^{n+\delta}}\,d\mu(y,t).$$

As before, if $(y,t)\notin T(2^jB)$, then $|y-z|+t\ge 2^j\sigma = c|2^jB|^{1/n}$. Thus, the above expression can be majorized by

$$c\sum_{j\geq 1}\frac{2^{j(\lambda/q-1/q-\delta/n)}}{|2^jB|^{\lambda/q-1/q+1}}\int_{T(2^{j+1}B)}|f(y,t)|\,d\mu(y,t)$$

$$\leq cC_{\lambda/q-1/q+1,1}(f,\mu)(x),$$

since $\lambda/q-1/q-\delta/n<0$. This completes the proof.

THEOREM (3.13). *Let $T_{k,\mu}$ be a Poisson-like operator, where μ is a Carleson measure of order α, $1\leq\alpha<\delta/n+1$. Then*

$$T_{k,\mu}:\ L_0^\infty(\mathbf{R}_+^{n+1})\to L_{q(\alpha-1),q}$$

continuously, provided $1<q<\infty$, $\alpha q'=p'$, and thus $\alpha<p'<\infty$.

Proof. If we replace the above hypothesis in Theorem (3.11), using Theorem (3.6)(iii), we conclude that (3.12) holds. Moreover,

$$\|C_{\lambda p/q,p}(f,\mu)\|_{L^\infty}\leq c\,\|\mu\|_{V^\alpha}^{1/p}\,\|f\|_{L^\infty},$$
$$\|C_{\lambda/q-1/q+1,1}(f,\mu)\|_{L^\infty}\leq c\,\|\mu\|_{V^\alpha}\,\|f\|_{L^\infty},$$

when $\alpha=\lambda p/q=\lambda/q-1/q+1$. Notice that $1\leq\alpha<\delta/n+1$ implies that $\delta/n-q(\lambda-1)>0$. Thus, according to (3.12), we conclude that

$$S_{q(\alpha-1)+1,q}(T_{k,\mu}(f))\in L^\infty.$$

This completes the proof of the theorem.

Remark (3.14). The conditions $1\leq\alpha<\delta/n+1$, $\alpha=(\lambda-1)/q+1$, $\alpha q'=p'$, $\alpha=\lambda p/q$ imply that $1\leq\lambda<\delta q/n+1$.

Thus, we deduce from the characterizations in Remarks (3.10)(i) that under the hypothesis in Theorem (3.13),

$$\begin{aligned}&T_{k,\mu}:\ L_0^\infty(\mathbf{R}_+^{n+1})\to \mathrm{BMO} && \text{if } \alpha=1,\\ &T_{k,\mu}:\ L_0^\infty(\mathbf{R}_+^{n+1})\to \mathrm{Lip}_{n(\alpha-1)} && \text{if } 1<\alpha<\delta/n+1.\end{aligned}\tag{3.15}$$

These continuity properties (3.15) can be seen in some cases as reinterpretations of known results on balayages (cf. [6]). Indeed, if μ is a finite Carleson measure and $f\in L^\infty(\mathbf{R}_+^{n+1})$, then $f d\mu$ is a finite Carleson measure and the balayage with a Poisson-like kernel is in BMO $\cap\, L^1$ (cf. [6], p. 31). Now, if $f\in L^\infty(\mathbf{R}_+^{n+1})$ and μ is a finite Carleson measure of order α, $1<\alpha<\delta/n+1$, then the balayage of $f d\mu$ with a Poisson-like kernel belongs to $L^\infty\cap \mathrm{Lip}_{n(\alpha-1)}$ (cf. [6], p. 31).

In the same spirit, we can prove

(3.16) $T_{k,\mu}:\ T_\infty^p\to L^p$ if μ is a Carleson measure, $1\leq p<\infty$.

Indeed, if $f\in T_\infty^1$, then by duality (cf. [13]), $f d\mu$ is a finite measure and therefore, its balayage is an integrable function (cf. [6]). Finally, (3.16) comes from interpolating $T^1_\infty \to L^1$ and $L^\infty(\mathbf{R}^{n+1}_+) \to \mathrm{BMO}$ (cf. [4], p. 159). In a similar fashion, we also have

$$(3.17)\qquad \begin{aligned} T_{k,\mu} &: T^{1/\alpha,1}_\infty \to L^{1/\alpha,1} \quad \text{if } \mu \in V^\alpha, \quad 0 < \alpha \le 1, \\ T_{k,\mu} &: T^{1/\alpha}_\infty \to L^{1/\alpha} \quad \text{if } \mu \in W^\alpha, \quad 0 < \alpha < 1. \end{aligned}$$

THEOREM (3.18). *Let $T_{k,\mu}$ be a Poisson-like operator, where μ is a Carleson measure of order $\alpha \ge 1$. Then $T_{k,\mu} : T^r_\infty \to L^s$ in each of the following cases:*

(i) $0 < r, s \le 1$, $1 \le \alpha \le 1/r$, $1/s = 1/r - \alpha + 1$,
(ii) $0 < r \le 1$, $1 < s < \infty$, $1/r < \alpha < 1/r + 1$, $1/s = 1/r - \alpha + 1$,
(iii) $0 < r \le 1$, $1/r + 1 = \alpha > 1 + \delta/n$, $s = \infty$.

Proof. (i) Pick a pair (p, q) satisfying the conditions in Theorem (3.6) (iii).

Now, given an (r, ∞) atom $a \in T^r_\infty$, suppose that $\mathrm{supp}(a) \subset T(B)$, B being the ball $B(z, \sigma)$. We can write

$$(3.19)\qquad \int |T_{k,\mu}(a)|^s\, dx = \int_{2B} |T_{k,\mu}(a)|^s\, dx + \int_{\mathbf{R}^n \setminus 2B} |T_{k,\mu}(a)|^s\, dx.$$

The first term can be majorized by

$$(3.20)\qquad c|B|^{1-s/q}\Big(\int |T_{k,\mu}(a)|^q\, dx\Big)^{s/q}$$
$$\le c|B|^{1-s/q}\Big(\int_{T(B)} |a(y,t)|^p\, d\mu(y,t)\Big)^{s/p} \le c|B|^{1-s/q+s/p-s/r} = c.$$

For the second term, we need to use condition (C_δ). Thus, we have

$$c \int_{\mathbf{R}^n \setminus 2B} \Big(\int_{T(B)} |k(x,y,t)|\, |a(y,t)|\, d\mu(y,t)\Big)^s dx$$
$$\le c|B|^{-s/r} \int_{\mathbf{R}^n \setminus 2B} \Big(\int_{T(B)} \frac{t^\delta}{(|x-y|+t)^{n+\delta}}\, d\mu(y,t)\Big)^s dx.$$

Now, if $x \notin 2B$ and $(y,t) \in T(B)$, then $|x - z| \ge 2\sigma > 2|y - z|$ and thus, $|x - y| + t \ge |x - z| - |y - z| + t > \frac{1}{2}|x - z|$. Thus, the integral above can be majorized by

$$c|B|^{\delta s/n - s/r + \alpha s} \int_{\mathbf{R}^n \setminus 2B} \frac{dx}{|x - z|^{(n+\delta)s}} \le c|B|^{\delta s/n - s/r + \alpha s + 1 - (1+\delta/n)s} = c.$$

This completes the proof of (i).

(ii) With the above notation, the estimate of the second term in (3.19) works the same. To majorize the first term, first choose $1 < p < \infty$ such that $\alpha < p' < \alpha s'$. Then the pair (p,q) with $q' = p'/\alpha$ will again satisfy the conditions in Theorem (3.6) (iii) and also, $q > s$. Thus, we can use Hölder's inequality with exponent q/s to obtain the same conclusion as in (3.20). This completes the proof of (ii).

(iii) Given an (r,∞) atom $a \in T^r_\infty$, suppose that $\operatorname{supp}(a) \subset T(B)$, B being the ball $B(z,\sigma)$. The conclusion will be an immediate consequence of the following pointwise inequality:

$$|T_{k,\mu}(a)(x)| \le c\chi_{2B}(x) + \frac{c}{|x-z|^{n+\delta}}|B|^{1+\delta/n}\chi_{\mathbf{R}^n\setminus 2B}(x). \tag{3.21}$$

Let us prove (3.21). We have

$$|T_{k,\mu}(a)(x)| \le c \int_{T(B)} \frac{t^\delta}{(|x-y|+t)^{n+\delta}}|a(y,t)|\,d\mu.$$

Suppose first that $x \in 2B$. Thus, $B \subset B(x,3\sigma)$ and

$$|T_{k,\mu}(a)(x)| \le c|B|^{\delta/n-1/r} \int_{T(B(x,3\sigma))} \frac{d\mu(y,t)}{(|x-y|+t)^{n+\delta}}.$$

Since $n+\delta < n\alpha$, we can apply Lemma 2 in [6], p. 32, to obtain

$$|T_{k,\mu}(a)(x)| \le c|B|^{\delta/n-1/r+1/n(\alpha n-\delta-n)} = c$$

if $x \in 2B$. Now, if $x \in \mathbf{R}^n \setminus 2B$, $(y,t) \in T(B)$, then $|x-y|+t \ge |x-y| \ge |x-z| - |y-z| \ge \frac{1}{2}|x-z|$. Thus,

$$|T_{k,\mu}(a)(x)| \le \frac{c}{|x-z|^{n+\delta}}|B|^{\delta/n-1/r+\alpha}$$

if $x \in \mathbf{R}^n \setminus 2B$.

This completes the proof of the theorem.

It should be observed that the condition $\alpha > 1+\delta/n$ in Theorem (3.18) (iii) is still unclear to us, except for its technical role in our proof.

THEOREM (3.22). *Let $T_{k,\mu}$ be a Poisson-like operator, where μ is a Carleson measure of order $\alpha \ge 1$. Moreover, assume that the kernel $k(x,y,t)$ satisfies the cancellation condition*

$$\int_{\mathbf{R}^n} k(x,y,t)\,dx = 0, \qquad (y,t) \in \mathbf{R}^{n+1}_+. \tag{3.23}$$

Then $T_{k,\mu}$ maps continuously T^r_∞ into H^s, provided that

$$0 < r \le 1, \qquad 1/r - \delta/n < \alpha \le 1/r, \qquad 1/s = 1/r + 1 - \alpha. \tag{3.24}$$

Proof. Pick a pair (p,q) satisfying the conditions in Theorem (3.6) (iii). It suffices to prove that given an (r,∞) atom a supported in a tent $T(B)$ over a ball $B = B(z,\sigma)$, the image $T_{k,\mu}(a)$ is an (s,q,θ) molecule related to B, for some $n(q/s-1) < \theta < q(n+\delta) - n$. That is to say, we will prove that $T_{k,\mu}(a)$ satisfies the following conditions (cf. [3]):

(i) $\int |T_{k,\mu}(a)|^q\,dx \le c|B|^{1-q/s}$,
(ii) $\int |T_{k,\mu}(a)|^q |x-z|^\theta\,dx \le c|B|^{\theta/n+1-q/s}$,
(iii) $\int T_{k,\mu}(a)\,dx = 0$.

Let us prove first condition (i). We have

$$\int |T_{k,\mu}(a)|^q\,dx \le c\Big(\int_{T(B)} |a(y,t)|^p\,d\mu(y,t)\Big)^{q/p}$$
$$\le c|B|^{-q/r+\alpha q/p} = c|B|^{1-q/s}.$$

Now, consider (ii). We split the integral in two terms:

$$\int_{2B} |T_{k,\mu}(a)|^q|x-z|^\theta\,dx$$
$$\le c|B|^{\theta/n}\Big(\int_{T(B)} |a(y,t)|^p\,d\mu(y,t)\Big)^{q/p} \le c|B|^{\theta/n-q/s+\alpha q/p}$$
$$= c|B|^{\theta/n+1-q/s}.$$

To estimate the other term,

$$\int_{\mathbf{R}^n\setminus 2B} |T_{k,\mu}(a)|^q|x-z|^\theta\,dx, \tag{3.25}$$

we will obtain a pointwise estimate for $|T_{k,\mu}(a)(x)|$, when $|x-z| > 2\sigma$, similar to the second term of (3.21). Indeed,

$$|T_{k,\mu}(a)(x)| \le \int_{T(B)} |k(x,y,t)|\,|a(y,t)|\,d\mu(y,t)$$
$$\le c\int_{T(B)} \frac{t^\delta}{(|x-y|+t)^{n+\delta}}|a(y,t)|\,d\mu(y,t).$$

If $|x-z| > 2\sigma$ and $(y,t)\in T(B)$, then $|x-y|+t \ge |x-z|-|y-z|+t > \frac{1}{2}|x-z|$. Thus, the above integral can be majorized by

$$c|B|^{\delta/n-1/r+\alpha}/|x-z|^{n+\delta}.$$

Then, replacing in (3.25), we obtain the estimate

$$c|B|^{(\delta/n-1/r+\alpha)q+1+\theta/n-q(1+\delta/n)} = c|B|^{\theta/n+1-q/s}.$$

It is shown in [3] that any function satisfying (i) and (ii) is integrable.

Thus, condition (iii) makes sense. Let us prove it. Indeed,

$$\begin{aligned}\int_{T(B)} |a(y,t)|\Big(\int_{\mathbf{R}^n} |k(x,y,t)|\,dx\Big)\,d\mu(y,t) \\ \le c \int_{T(B)} |a(y,t)|\Big(\int_{\mathbf{R}^n} \frac{t^\delta}{(|x-y|+t)^{n+\delta}}\,dx\Big)\,d\mu(y,t) \\ = c\int_{T(B)} |a(y,t)|\,d\mu(y,t) \le c|B|^{\alpha-1/r}.\end{aligned}$$

Thus,

$$\begin{aligned}\int T_{k,\mu}(a)(x)\,dx &= \int\Big(\int k(x,y,t)a(y,t)\,d\mu(y,t)\Big)\,dx \\ &= \int a(y,t)\Big(\int k(x,y,t)\,dx\Big)\,d\mu(y,t) = 0.\end{aligned}$$

This completes the proof of the theorem.

Remark (3.26). The cancellation condition (3.23) is frequently satisfied. Indeed, the kernel of the operator π_φ considered in Example (2.2) satisfies (3.23); so does the kernel of the operator Q_tTP_t considered in Example (2.15), provided that $T^*(1) = 0$ in the sense of BMO (cf. [3]).

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Reçu par la Rédaction le 30.3.1990