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LINEARLY FUNDAMENTALLY ORDERED SEMIGROUPS

BY

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In this paper a *function on a set* A means a transformation of a subset of A into A. *Superposition* $g \circ f$ of two functions on A is defined as a function on A such that, for every $a \in A$, we have $g \circ f(a) = g(f(a))$, where the left-hand side of this equality is meaningful if the right-hand side is.

A *function semigroup* is a non-empty set of functions on a set A, closed with respect to the superposition which is then considered as the semigroup operation. Every function semigroup $\mathscr{F}$ is ordered by the inclusion relation: $f \subset g$ means that g is an extension of f.

Let S be a semigroup ordered by an order relation $\leqslant$ and P be a map of S onto a function semigroup $\mathscr{F}$. Then P is an order-isomorphism if and only if $s \leqslant t \leftrightarrow \mathsf{P}(s) \subset \mathsf{P}(t)$ and $\mathsf{P}(st) = \mathsf{P}(t) \circ \mathsf{P}(s)$ for all $s, t \in S$. The form of the last equality depends on the fact that in the product st the first factor is s, and in the product $\mathsf{P}(t) \circ \mathsf{P}(s)$ the first factor is $\mathsf{P}(s)$. The ordered semigroup $(S, \leqslant)$ is called *fundamentally ordered* if it is order-isomorphic with an inclusion-ordered function semigroup. $(S, \leqslant)$ is *strictly fundamentally ordered* if it is order-isomorphic with an inclusion-ordered semigroup of univalent (i.e., one-to-one) functions.

The aim of this paper is to give a complete description of all linearly fundamentally ordered semigroups, all linearly fundamentally orderable semigroups and all semigroups on which every fundamental order may be extended to a linear fundamental order. The analogous problem is solved for strict fundamental orders.

An order relation $\leqslant$ on a semigroup S is *stable* if

$$s_1 \leqslant s_2 \wedge t_1 \leqslant t_2 \to s_1 t_1 \leqslant s_2 t_2$$

(here $\wedge$ is the sign of conjunction, $\to$ is the sign of implication). An order relation $\leqslant$ is called *weakly steady* if

$$z \leqslant xv \wedge z \leqslant uy \wedge x \leqslant u \to z \leqslant xy$$

for all $u, x, z \in S$ and all $v, y \in S^1$; $\leqslant$ is called *steady* if

$$z \leqslant xv \wedge z \leqslant uv \wedge z \leqslant uy \to z \leqslant xy$$

for all $z \in S$ and $u, v, x, y \in S^1$ such that $xv, uv, uy, xy \in S$. Here S^1 denotes S in the case where S contains an identity, otherwise S^1 is S with an identity adjoined.

The main tool used in the present paper is the following

THEOREM ON FUNDAMENTAL ORDERS([1]). *An ordered semigroup is fundamentally ordered if and only if the order relation on the semigroup is stable and weakly steady; it is strictly fundamentally ordered if and only if the order is stable and steady.*

To formulate our main results we need the following constructions:

Construction I. Let I be a non-empty set linearly ordered by an irreflexive order relation $\prec$ and let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of non-empty sets indexed by I and such that

1° $A_i \cap B_j \neq \emptyset \leftrightarrow A_i \cap A_j \neq \emptyset \leftrightarrow B_i \cap B_j \neq \emptyset \leftrightarrow i = j$,

2° $A_i \cap B_i$ is a one-element set denoted by $\{a_i\}$.

Write $C_i = A_i \cup B_i$, $C = \bigcup (C_i)_{i\in I}$ and define a binary multiplication in C in the following way:

$$xy = \begin{cases} a_j & \text{if } x \in C_i, y \in C_j, j \prec i, \\ a_i & \text{if } x, y \in C_i, x \notin A_i, \\ x & \text{otherwise.} \end{cases}$$

One can verify that C endowed with this operation is a semigroup, C_i are subsemigroups of C, $A = \bigcup (A_i)_{i\in I}$ is the set of all idempotents of C and A is a subsemigroup of C.

Construction II. Let C be a semigroup described in Construction I. Let every set A_i be linearly ordered in such a way that a_i is the largest element of A_i. If i is the largest element of I and $B_i = \{a_i\}$, then the linear order on A_i may be arbitrary and a_i need not be the largest element of A_i. Let $B_i \setminus A_i$ be linearly ordered in an arbitrary way. There exists a uniquely determined linear order $\leqslant$ on C such that each element of C_i precedes each element of C_j for $i \prec j$, each element of A_i precedes each element of $B_i \setminus A_i$ for all $i \in I$, and restrictions of $\leqslant$ on A_i and $B_i \setminus A_i$ coincide with the order relations given on these sets for all $i \in I$.

Let us show that $(C, \leqslant)$ is a fundamentally ordered semigroup. To prove stability of $\leqslant$, suppose $x \in C_i, y \in C_j, z \in C_k$, and $x \leqslant y$. Then $i \prec j$ or $i = j$. If $k \prec i$, then $xz = a_k = yz$; if $i = k \prec j$, then $xz \in A_i$ and $yz = a_i$, hence, $xz \leqslant yz$; if $i \prec k \prec j$, then $xz = x \leqslant a_k = yz$; if $i \prec k = j$, then $xz = x \leqslant yz \in C_j$; if $j \prec k$, then $xz = x \leqslant y = yz$. Now let $i = j = k$. If $y \in A_i$, then $x \in A_i$ and $xz = x \leqslant y = yz$; if $y \notin A_i$, then $yz = a_i$ and $xz \in A_i$, whence $xz \leqslant yz$. Therefore, $x \leqslant y \to xz \leqslant yz$, i.e., $\leqslant$ is right regular.

([1]) Б. М. Шайн, *Представление упорядоченных полугрупп*, Математический сборник 65 (1964), p. 188 - 197.

Now if $k \prec i$, then $zx = z = zy$; if $i \prec k \prec j$, then $zx = a_i \leqslant z = zy$; if $i \prec k = j$, then $zx = a_i \leqslant zy \in C_j$; if $j \prec k$, then $zx = a_i \leqslant a_j = zy$. If $i = k$ and $z \in A_i$, then $zx = z = zy$; if $i = k$ and $z \notin A_i$, then $zx = a_i$ and zy equals either z or a_i — in both cases $zx \leqslant zy$. Thus, $\leqslant$ is right and left regular, which means $\leqslant$ is stable.

To prove the weak steadiness of $\leqslant$, suppose $z \leqslant xv$, $z \leqslant uy$ and $x \leqslant u$ for $u, x, z \in C$ and $v, y \in C^1$. One can easily see that $xv \leqslant x$, whence $z \leqslant x$ and $zy \leqslant xy$. Suppose $z \in C_i$ and $y \in C_j$. If either $j \prec i$ or $i = j$ and $z \in A_i$, then $uy \leqslant a_i < z \leqslant uy$, a contradiction. It follows that either $i \prec j$ or $i = j$ and $z \in A_i$. In both cases $z = zy \leqslant xy$.

Thus $(C, \leqslant)$ is a linearly fundamentally ordered semigroup. Now we are able to formulate the main results:

THEOREM 1. *A semigroup can be linearly fundamentally ordered if and only if it belongs to the class of semigroups built up in Construction* I. *These semigroups are precisely those which satisfy the following elementary universal conditions*:

I. $xy^2 = xy$;

II. $xyz = xzy$;

III. $x^2y = x^2 \vee y^2x = y^2$;

IV. $xy = x^2 \wedge yx = y^2 \to x = x^2 \vee y = y^2 \vee x^2 = y^2$;

V. $zx = z \wedge z^2y = z \to x = x^2 \vee y = y^2 \vee xz = yz$;

VI. $x^2y = x^2 \to yx = y^2 \vee x^2 = x \vee (xy = x \wedge yx = x^2)$.

Here $\vee$ is the sign of disjunction; conjunction and disjunction tie up the formulas stronger than implication does: $\alpha \wedge \beta \to \gamma \vee \delta$ means $(\alpha \wedge \beta) \to (\gamma \vee \delta)$.

THEOREM 2. *The class of all linearly fundamentally ordered semigroups coincides with the class of ordered semigroups built up in Construction* II.

COROLLARY. *A semigroup S can be linearly fundamentally ordered in a unique way if and only if S satisfies conditions* I—VI, *S has at most one right identity and every left zero subsemigroup and every zero subsemigroup of S have at most two elements.*

THEOREM 3. *Every fundamental order on a semigroup S can be extended to a linear fundamental order if and only if S has the structure described in Construction* I *and every left zero subsemigroup of S either is trivial or consists of right identities of S.*

THEOREM 4. *An ordered semigroup $(S, \leqslant)$ is a linearly strictly fundamentally ordered semigroup if and only if $S = A \cup B$, where A is a linear semilattice and B is either empty or a zero semigroup. Every element of B acts as identity on A, the order $\leqslant$ coincides on A with the natural semilattice order, all the elements of A are smaller than all the elements of B, the zero of B is the smallest element of B.*

Proofs. Let $(S, \leqslant)$ be a linearly fundamentally ordered semigroup. We shall prove that S satisfies conditions I—VI. If $x \leqslant xy$, then $xy \leqslant x \cdot y$, $xy \leqslant xy \cdot 1$, $x \leqslant xy$, whence, by the weak steadiness of $\leqslant$, $xy \leqslant x \cdot 1 = x$ and $xy = x$. Thus, $xy \leqslant x$ for all $x, y \in S$. Suppose now $x \leqslant yz$. Since $yz \leqslant y$, we obtain $x \leqslant x$, $x \leqslant y \cdot z$, and $x \leqslant y$. By the weak steadiness of $\leqslant$, $x \leqslant xz \leqslant x$. Thus, $x \leqslant yz \to x = xz$.

The just proved two formulas

$$xy \leqslant x, \qquad x \leqslant yz \to x = xz$$

will be used later on without special reference.

Now $xy \leqslant xy$ implies $xy = xyy = xy^2$, $xyz = x(yz)^2 = xyzyz \leqslant xzyz \leqslant xzy$. In the same way, $xzy \leqslant xyz$. Thus, $xyz = xzy$. For all x and y there is $x^2 \leqslant y^2$ or $y^2 \leqslant x^2$. It follows $x^2y = x^2 \vee y^2x = y^2$. If $xy = x^2$ and $yx = y^2$, then $x \leqslant y^2$ implies $x^2 = xy = x$, $y \leqslant x^2$ implies $y^2 = yx = y$, $y^2 < x \wedge x^2 < y$ implies $y^2 = y^3 \leqslant xy = x^2$ and, in the same way, $x^2 \leqslant y^2$, whence $x^2 = y^2$.

Let $zx = z^2y = z$. Then $z = z^2y = z^3y = z^2$. If $x \leqslant z$, then $x \leqslant z^2$ and $xz = x$. Therefore, $x^2 = (xz)^2 = xzxz = xz^2x = xz = x$. In the same way, $y \leqslant z \to y = y^2$. One can easily verify that $(xz)^2 = xz$ and $(yz)^2 = yz$. In the same way as above, $y \leqslant xz \to y = y^2$, $x \leqslant yz \to x = x^2$. If $xz \leqslant y$ and $yz \leqslant x$, then $xz = xz^2 \leqslant yz$ and $yz = yz^2 \leqslant xz$. This, $xz = yz$. Condition V is verified.

Now let $x^2y = x^2$. If $y^2 \leqslant x$, then $y^2 = y^4 \leqslant x^2$ and $y^2 = y^2x = yxy = yx^2y = yx^2 = yx$; if $x \leqslant yx$, then $x = x^2$, if $yx \leqslant x \leqslant y^2$, then $xy = x$ and $yx = yx^2 \leqslant x^2 \leqslant y^2$ $x = yxy = yx$ and $yx = x^2$. Thus, VI is verified.

To prove that $(S, \leqslant)$ has the structure described in Construction II, define a binary relation ϱ on S: $(x, y) \in \varrho \leftrightarrow x^2y = x^2$. By I, ϱ is reflexive, $x^2y = x^2$ and $y^2z = y^2$ imply $x^2z = x^2yz = x^2y^2z = x^2y^2 = x^2y = x^2$, thus ϱ is transitive, i.e., ϱ is a quasi-order relation. Let ε be the symmetric part of ϱ. Then the quotient set $S/\varepsilon = I$ is linearly (by III) ordered by ϱ/ε. Let $\prec$ be the irreflexive part of ϱ/ε and C_i be the ε-class corresponding to $i \in I$. Let $x \leqslant y \leqslant z$ and $x, z \in C_i$. Then $x^2 = x^2x \leqslant x^2y \leqslant x^2z = x^2$ and $x^2y = x^2$, and, on the other hand, $y^2 = y^2y \leqslant y^2z \leqslant y^2$ and $y^2z = y^2$. Therefore, $y \in C_i$ and C_i is convex. Now $y \leqslant z \to (y, z) \in \varrho$, as we have proved. It follows that S is the ordinal sum of $(C_i)_{i \in I}$ along $(I, \prec)$, each C_i ordered by the restriction of $\leqslant$. If $x, y \in C_i$, then $xy = xy^2 = xy^2x = xyx = x^2 \in C_i$, thus C_i is subsemigroup of S. Let $x \in C_i$, $x \in C_j$, $i \prec j$. Then $x^2y = x^2$ and $y^2x \neq y^2$. If $yx = y^2$, then $y^2x = yy^2 = y^2$, whence $yx \neq y^2$ and, by VI, $xy = x$. Let A_i be the set of all idempotents in C_i. If $x \in A_i$ and $y \in C_i$, then $xy = x^2y = x^2 = x$, if $y \leqslant x$, then $y \leqslant x^2$ and $yx = y$. It follows that $y^2 = yxyx = yx^2y = yxy = yx = y$. Thus $y \in A_i$, C_i is the ordinal sum of A_i, and $C_i \setminus A_i$ ordered by the restrictions of $\leqslant$. If $x, y \in C_i \setminus A_i$, then, by IV, $x^2 = y^2 \in A_i$ and $x^2 = x^3 = xy^2 = xy$. The

common value of x^2, y^2, and xy will be denoted as a_i. Let $B_i = (C_i \setminus A_i) \cup \{a_i\}$. If $x \in B_i$ and $y \in A_i$, then $y = yx$ and $xy = xyx = x^2y = x^2 = a_i$, thus $B_iC_i = B_iB_i \cup B_iA_i = \{a_i\}$; moreover, $y \leqslant x$ and $y = y^2 \leqslant x^2 = a_i$, i.e. a_i is the largest element of A_i. Now let $x \in C_i$, $y \in C_j$, $i \prec j$. Then $x^2y = x^2$ and $y^2x \neq y^2$. If $yx = y^2$, then, by I, $y^2 = yy^2 = yyx \neq y^2$, a contradiction. By VI, $x \in A_i$ or $yx = x^2 = a_i$.

Let $x \in A_i$. Then $(yx)^2x = (yx)^2$ and $x^2(xy) = x^3y = x^2y = x^2$, and therefore $yx \in C_i$. Since $(yx)^2 = yxyx = yx^2y = yx^2 = yx$, $yx \in A_i$. If $z \in A_i$, then $z \leqslant y$ and $z = zx \leqslant yx$. It follows that yx is the largest element of A_i. If the element a_i has been defined, then $yx = a_i$, otherwise let us define $yx = a_i$. Now a_i is defined for all $i \in I$ except the case when i is the largest element of I and $A_i = C_i$. This being the case, define a_i arbitrarily. We have proved that $(S, \leqslant)$ has the structure described in Construction II and S has the structure described in Construction I. On the other hand, every semigroup described in Construction I is linearly fundamentally orderable (see Construction II) and every semigroup described in Construction II is linearly fundamentally ordered.

To end the proof of theorems 1 and 2 we need to verify that every semigroup satisfying conditions I—VI is linearly fundamentally orderable.

Suppose S satisfies I—VI. Define ϱ and ε as above. Then $S = \bigcup (C_i)_{i \in I}$ and I is linearly ordered by $\prec$. The same argument as above shows that $C_i = A_i \cup B_i$, where $B_i = (C_i \setminus A_i) \cup \{a_i\}$, a_i being a special element of A_i defined by axioms I—VI. Now we can see that S is a semigroup built up along the plan exposed in Construction I. Therefore S is linearly fundamentally orderable.

To prove the corollary note that, as follows from Construction II and Theorem 2, S can be linearly fundamentally ordered in a unique way if and only if $|A_i| \leqslant 2$, $|B_i| \leqslant 2$ and if i is the largest element of I and $C_i = A_i$, then $|A_i| = 1$ (here $|A|$ denotes the cardinality of A). These conditions mean that every left zero subsemigroup and every zero subsemigroup (which are included in an A_i and a B_j, respectively) have at most two elements; the set of right identities of S is just the set A_i, where i is the largest element of I and $C_i = A_i$. Thus S contains at most one right identity.

To prove Theorem 3 note that the identical order relation ($x \leqslant y \leftrightarrow x = y$) is fundamental, and so every semigroup with extendable fundamental orders can be linearly fundamentally ordered and, by Theorem 1, satisfies I—VI.

Now let S be a semigroup satisfying I—VI. By Theorem 1, S has the structure described in Construction I. Let $\leqslant$ be a fundamental order on S. By Construction II and Theorem 2, $\leqslant$ may be extended to a linear fundamental order if and only if for all $x \in C_i$, $y \in C_j$, $x \leqslant y$ the following conditions are satisfied:

1) $i \neq j \to i \prec j$;

2) $i = j \wedge y \in A_i \to x \in A_i$;

3) $i = j$, $x = a_i$, and $y \in A_i$ is possible only if i is the largest element of I and $A_i = C_i$.

Thus we have to prove conditions 1)—3). If $u \leqslant uv$, then $uv \leqslant u \cdot v$, $uv \leqslant uv \cdot 1$, and $u \leqslant uv$. By the weak steadiness of $\leqslant$, $uv \leqslant u \cdot 1 = u$. Therefore, $u \leqslant uv \to u = uv$.

1) Let $i \neq j$. Then $x^2 \leqslant y^2$, and so $x^2 = x^2x^2 \leqslant x^2y^2 = x^2y$. It follows that $x^2 = x^2y$ which means that $i \prec j$.

2) Let $i = j$ and $y \in A_i$. Then $x \leqslant x \cdot 1$, $x \leqslant y \cdot y = y$, and $x \leqslant y$. By the weak steadiness of $\leqslant$, $x \leqslant x \cdot y = xy^2 = xy^2x = x^2y^2 = x^2y = x^2$. Hence, $x = x^2$.

3) Define the following order relation $\leqslant$ on S: $x \leqslant y$ means either $x = y$ or $x = a_i \wedge y \in A_i$ for some $i \in I$. Clearly, if $x \leqslant y$, then $x, y \in C_i$ for some i; therefore $zx = zx^2 = zx^2y = zxy = zyx = zy^2x = zy^2 = zy$ and $x \leqslant y \to zx \leqslant zy$. If $x = y$, then $xz = yz$. If $x < y$, then $x = a_i$ and $y \in A_i$; one can easily verify that $xz \leqslant yz$ in this case. Thus, $\leqslant$ is stable.

To show $\leqslant$ is weakly steady, suppose $z \leqslant xv$, $z \leqslant uy$, $x \leqslant u$. If $x = u$, then $z \leqslant uy = xy$. Let $x < u$. Then $x = a_i$ and $u \in A_i$. If $y = 1$, then either $z = u$ or $z = a_i$. In both cases $z \leqslant xv$ implies $xv = x$, i.e., $z \leqslant a_i$. Therefore, $z = a_i$ and $z \leqslant xy$. The same argument holds if $uy = u$.

Now let $uy \neq u$. Then $y \in C_j$, $j \prec i$, and $uy = a_j$. Thus $z \leqslant a_j$. It follows that $z = a_j$, $xy = a_j$, and $z = xy$.

Thus, if $|A_i| \neq 1$ for some i which is not the largest element of I or such that $C_i \neq A_i$, then $\leqslant$ cannot be extended to a linear fundamental order. Hence conditions of Theorem 3 are necessary. Sufficiency follows from the fact that every semigroup satisfying conditions of Theorem 3 satisfies the above-mentioned condition 3).

The structure of semigroups described by Theorem 3 is the following: let $(I, \prec)$ be a linearly irreflexively ordered set, and let $(B_i)_{i \in I}$ be a family of zero semigroups, a_i being the zero of B_i. Let $B = \bigcup (B_i)_{i \in I}$. Define the following operation in B: if $x \in B_i$, $y \in B_j$, then $xy = x$ in the case $i \prec j$ and $xy = a_j$ otherwise. Let L be a left zero semigroup (L may be empty). Define $S = L \cup B$ and define the binary operation in S as follows: the operation acts in L and in B as the operations of semigroups L and B, if $x \in L$ and $y \in B$, then $xy = y^2$, $yx = y$. The class of just defined semigroups coincides with the class of semigroups on which every fundamental order may be extended to a linear fundamental order.

To prove Theorem 4, suppose $(S, \leqslant)$ is a linearly strictly fundamentally ordered semigroup. Since every strict fundamental order is a fundamental order, $(S, \leqslant)$ has the structure described in Construction II. In the proof of Theorems 1 and 2 we have seen that $xy \leqslant x$ for all $x \in S$. If S^* is the dual semigroup relative to S (i.e., $xy = z$ in S^* if and only if $yx = z$

in S), then also $(S^*, \leqslant)$ is a linearly strictly fundamentally ordered semigroup. Hence $xy \leqslant y$ for all $x, y \in S$. If x and y are idempotents, then also xy is. We obtain $xy = (xy)^2 \leqslant yx$. In the same way $yx \leqslant xy$. Thus $xy = yx$ and the idempotents of S commute. It follows that $|A_i| = 1$ for all $i \in I$.

Suppose $x \leqslant y = y^2$. Then $x \leqslant x \cdot 1$, $x \leqslant y \cdot 1$, $x \leqslant y \cdot y$. By the steadiness of $\leqslant$, $x \leqslant xy$. In the same way, $x \leqslant yx$. Therefore $x \leqslant x \cdot y$, $x \leqslant y \cdot y$, $x \leqslant y \cdot x$ and, by the steadiness of $\leqslant$, $x \leqslant xx = x^2$. Thus $x = x^2$.

If $x \in C_i$, $y \in A_j$, and $i \rightarrowtail j$, then $x \leqslant y = y^2$. Thus $x^2 = x \in A_i$, whence $C_i = \{a_i\}$.

It follows that S has the structure described in Theorem 4. Evidently, every semigroup with such a structure is linearly strictly fundamentally ordered.

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