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SPLITTING HOMOMORPHISMS AND BOUNDED 3-MANIFOLDS

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In this paper we investigate the relationships between bounded, compact 3-manifolds and homomorphisms of the fundamental groups of bounded, compact surfaces. We show that, modulo the conjecture that the algebraic and topological mapping class groups of a 2-manifold are isomorphic, 3-manifolds (compact and bounded) correspond, in a natural fashion, to certain homomorphisms of the fundamental groups of certain embedded surfaces.

In [6] Stallings defined the notion of a splitting homomorphism for the fundamental group of a closed, orientable surface, and Jaco [3] has shown that there is a bijection from the equivalence classes of such homomorphisms to equivalence classes of Heegaard splittings of closed, orientable 3-manifolds. It is our purpose to define splitting homomorphisms for the fundamental groups of compact, orientable, bounded surfaces and to show that they correspond similarly to splittings of compact, orientable, bounded 3-manifolds if and only if certain types of automorphisms of 2-manifold groups can be represented as homeomorphisms.

We will work entirely in the piecewise linear category, so all spaces are simplicial complexes and all maps are piecewise linear.

Let S be a compact, orientable surface of genus n with k boundary components. Let F_1 and F_2 be free groups of rank $n+k-p$ $(p \leqslant k)$ and let $F_1 \times F_2$ denote their direct product. Fix a point $s_0 \in \partial S$. Now let

$$S' = S \cup (D_1 \cup \ldots \cup D_p),$$

where D_t is a 2-sphere with holes, the total number of holes is k,

$$D_t \cap S = \partial D_t \cap \partial S = \partial D_t \quad \text{for each } t,$$

and S' is a closed, orientable surface of genus $n+k-p$. Suppose $j\colon S \to S'$ is inclusion and $f_i\colon \pi_1(S, s_0) \to F_i$ is a surjective homomorphism for $i = 1, 2$.

For any $q \leqslant p$, let a_q be a simple path in S from s_0 to a point $x_q \in D_q$. Let

$$a_q \pi_1(D_q, x_q) a_q^{-1} = \{[a_q * g * a_q^{-1}] \mid [g] \in \pi_1(D_q, x_q)\}.$$

Then $a_q\pi_1(D_q, x_q)a_q^{-1}$ is a subgroup of $\pi_1(S', s_0)$. The homomorphism

$$(f_1, f_2)\colon \pi_1(S, s_0) \to F_1 \times F_2$$

defined by

$$(f_1, f_2)[g] = (f_1[g], f_2[g])$$

is called a *splitting homomorphism* of $\pi_1(S, s_0)$ provided there are a homomorphism $(f_1', f_2')\colon \pi_1(S', s_0) \to F_1 \times F_2$ and a choice of paths $a_1, \ldots, a_p$ such that $(f_1', f_2')\circ j = (f_1, f_2)$, and the subgroups $f_i'\big(a_q\pi_1(D_q, x_q)a_q^{-1}\big)$ are free factors of F_i $(i = 1, 2)$ which are pairwise disjoint except for the identity element.

A splitting homomorphism $(v_1, v_2)\colon \pi_1(R, r_0) \to H_1 \times H_2$ is said to be *equivalent to* (f_1, f_2) if there are isomorphisms $a\colon \pi_1(S, s_0) \to \pi_1(R, r_0)$ and $a_i\colon F_i \to H_i$ $(i = 1, 2)$ such that the diagram

$$\begin{tikzcd}
\pi_1(S, s_0) \arrow[r, "(f_1, f_2)"] \arrow[d, "a"] & F_1 \times F_2 \arrow[d, "a_1 \times a_2"] \\
\pi_1(R, r_0) \arrow[r, "(v_1, v_2)"] & H_1 \times H_2
\end{tikzcd}, \quad \text{where } (a_1 \times a_2)(x, y) = \big(a_1(x), a_2(y)\big),$$

commutes. The equivalence is called *homeomorphism induced* if there is a homeomorphism $h\colon (S, s_0) \to (R, r_0)$ such that $h_* = a$.

Let M be a compact, orientable 3-manifold with non-empty boundary. It is known [2] that M has a *D-splitting*, that is, $M = U_1 \cup U_2$, where U_1 and U_2 are solid tori of the same genus, $U_1 \cap U_2 = \partial U_1 \cap \partial U_2 = S$ is a connected surface, the inclusion of S into each of U_1 and U_2 induces surjections of the fundamental group, and the inclusion of each component of $\partial U_i - S$ induces an injection of its fundamental group onto a free factor of $\pi_1(U_i)$ so that no two images meet in more than the identity element. Moreover, we may assume that each component of $\partial U_i - S$ is planar.

Let $N = P_1 \cup P_2$ be a D-splitting of the 3-manifold N. Then the splitting $P_1 \cup P_2$ is *equivalent to* $U_1 \cup U_2$ if there is a homeomorphism $h\colon N \to M$ such that $h(P_i) = U_i$ $(i = 1, 2)$.

If $s_0 \in \partial S$, then the inclusion of S into U_1 and U_2 induces surjections $u_{i*}\colon \pi_1(S, s_0) \to \pi_1(U_i, s_0)$. The map (u_{1*}, u_{2*}) is a splitting homomorphism and is called the *splitting homomorphism induced by* $U_1 \cup U_2$. Clearly, equivalent D-splittings induce equivalent splitting homomorphisms.

By a simple modification of a proof of Stallings [6], we get

THEOREM 0. *With the above notation* $\pi_1(M, s_0)$ *is isomorphic to the quotient group of* $\pi_1(S, s_0)$ *by* $(\operatorname{kernel} u_{1*})\cdot(\operatorname{kernel} u_{2*})$.

THEOREM 1. *Let* $(f_1, f_2)\colon \pi_1(S, s_0) \to F_1 \times F_2$ $(\operatorname{rank} F_i = n + k - p)$ *be a splitting homomorphism of* $\pi_1(S)$, *where* S *has genus* n *and* k *boundary* *components. Then there are a compact 3-manifold M with p boundary components and a D-splitting $U_1 \cup U_2$ of M such that the induced splitting homomorphism is equivalent to (f_1, f_2).*

To see this, let $S' = S \cup (Q_1 \cup Q_2)$, where Q_i $(i = 1, 2)$ is a collection of 2-spheres with holes such that $Q_1 \cap Q_2 = Q_i \cap S = \partial Q_i = \partial S$, and $S_i' = S \cup Q_i$ is an orientable surface of genus $n+k-p$. Then, following the proof of Theorem 5.2 in [3], we find functions g_i' mapping S_i' onto a wedge of circles such that the union of the mapping cylinders $G(g_1') \cup G(g_2')$ is a D-splitting for a manifold M, $(g_i)_* = f_i$, where $g_i = g_i'|S$, and the induced splitting homomorphism is equivalent to (f_1, f_2).

THEOREM 2. *Suppose $M = U_1 \cup U_2$ and $N = P_1 \cup P_2$ are D-splittings that induce equivalent splitting homomorphisms. Then $M = U_1 \cup U_2$ is equivalent to $N = P_1 \cup P_2$ if and only if some equivalence of splitting homomorphisms is homeomorphism induced.*

Proof. Let $S = \partial U_1 \cap \partial U_2$ and $R = \partial P_1 \cap \partial P_2$. Then there are points $s_0 \in \partial S$ and $r_0 \in \partial R$ and isomorphisms

$$a\colon \pi_1(S, s_0) \to \pi_1(R, r_0), \qquad a_i\colon \pi_1(U_i, s_0) \to \pi_1(P_i, r_0) \ (i = 1, 2)$$

such that the diagram

$$\begin{tikzcd} \pi_1(S, s_0) \arrow[r, "u_{i*}"] \arrow[d, "a"] & \pi_1(U_i, s_0) \arrow[d, "a_i"] \\ \pi_1(R, r_0) \arrow[r, "p_{i*}"] & \pi_1(P_i, r_0) \end{tikzcd}$$

commutes, where all other homomorphisms are induced by inclusion. Assume a is homeomorphism induced by $h\colon (S, s_0) \to (R, r_0)$ so that $h_* = a$. We now need only to show that h can be extended to a homeomorphism from (U_i, s_0) onto (P_i, r_0) $(i = 1, 2)$.

We choose [7] *cutting disks* $G_1, \ldots, G_{n+k-p}$ of U_1 (that is, properly embedded, pairwise disjoint disks such that the result of cutting U_1 along $\{G_i\}$ is a 3-cell) so that

(1) $G_j \cap S$ is an arc for $j = 1, \ldots, k-p$,

(2) $G_j \cap S = \partial G_j$ for $j = k-p+1, \ldots, k-p+n$, and

(3) if K^0 is a component of $\partial U_1 - S$, then $K^0 - \bigcup G_i$ is an open disk.

Let K be the closure of a component of $\partial U_1 - S$ and suppose $G_1, \ldots, G_\mu$ are the disks that meet K. (Note that K is a disk with μ holes.) Let $J_i = \partial G_i \cap S$ $(i = 1, \ldots, \mu)$ and let E_i be a regular neighborhood of J_i in S that meets ∂S in two arcs. Now $\partial\big(K \cup (\bigcup E_i)\big)$ is a simple closed curve in S that bounds a properly embedded disk A in U_1. Then, by commutativity of the diagram and the fact that a is induced by the homeomorphism h, $h\Big(\partial\big(K \cup (\bigcup E_i)\big)\Big)$ is null homotopic in P_1. Hence, by Dehn's lemma [5], and since it is a simple closed curve, it bounds a non-singular, properly embedded disk A' in P_1.

If $\partial U_1 - S$ is connected, then so is $\partial P_1 - R$, and $h(\partial K)$ bounds $K' = \mathrm{Cl}(\partial P_1 - R)$. So suppose $\partial U_1 - S$ is not connected. Let $B_1, \ldots, B_v$ be the boundary components of K. We can choose canonical, geometric free generators for $H_1(S)$ and $H_1(U_1)$ that contain $\{B_1, \ldots, B_v\}$ and $\{B_2, \ldots, B_v\}$, respectively, with $B_1 + \ldots + B_v = 0$ in $H_1(U_1)$. Then, since $a = h_*$ and the above-given diagram is commutative, we infer that $\{h(B_1), \ldots, h(B_v)\}$ and $\{h(B_2), \ldots, h(B_v)\}$ lie in sets of geometric free generators for $H_1(R)$ and $H_1(P_1)$, respectively, and that $h(B_1) + \ldots + h(B_v) = 0$ in $H_1(P_1)$. But now $\{h(B_2), \ldots, h(B_v)\}$ forms a set of $v-1$ geometric free generators for $\pi_1(P_1)$ and the fundamental groups of components of $\partial P_1 - R$ embed onto disjoint (except for the identity) free factors of $\pi_1(P_1)$, so $h(B_1), \ldots, h(B_v)$ must all lie in the boundary of the same component of $\partial P_1 - R$, whose closure we denote by K'. Since $h(B_1) + \ldots + h(B_v) = 0$ in $H_1(P_1)$, we have $\partial K' = \bigcup h(B_i) = h(\partial K)$. Hence, $K' \cup A' \cup h\big(\bigcup E_i\big)$ is an orientable surface of genus μ in P_1 and bounds a solid torus P^* of genus μ in P_1. Furthermore, $\partial P^* - \partial P_1 = \mathrm{Int}\, A'$ and $\bigcup h(J_i) \subset P^* \cap R$. Choose now pairwise disjoint arcs $J_1', \ldots, J_\mu'$ in $P^* - \mathrm{Int}\, A'$ such that

(1) $J_i' \cap J_j' = \emptyset$ if $i \neq j$,

(2) $h(J_i) \cap J_i' = \partial h(J_i) = \partial J_i'$, and

(3) each $h(J_i) \cup J_i'$ bounds a properly embedded disk C_i in P^* so that $\{C_1, \ldots, C_\mu\}$ is a complete set of cutting disks for P^*.

We now repeat this process for the other components of $\partial U_1 - S$ being careful that the solid tori we construct in P_1 are pairwise disjoint. Also, by Dehn's lemma [5] and the loop theorem [4], $h(\partial G_j)$ $(j \geqslant k-p+1)$ bounds a disk C_j in P_1. These may be taken by standard disk replacement to be pairwise disjoint and disjoint from all P^*. We now may extend h in the obvious fashion so that $h(G_i) = C_i$ $(i = 1, \ldots, n+k-p)$, and then to all of U_i, since $\{G_1, \ldots, G_{n+k-p}\}$ is a set of cutting disks for U_1, and $\{C_1, \ldots, C_{n+k-p}\}$ is a set of cutting disks for P_1.

We repeat the process to extend h to a homeomorphism from U_2 onto P_2. This then induces a homeomorphism $h\colon M \to N$ such that $h(U_1) = P_1$, and $h(U_2) = P_2$.

The converse statement follows immediately from our definitions, so the proof is complete.

The question of whether or not all splitting homomorphism equivalences are homeomorphism induced is closely related to, but weaker than, the conjecture in [1] that the algebraic mapping class group is isomorphic to the topological mapping class group for 2-manifolds. Therefore, we ask the following:

(1) If some equivalence between two splitting homomorphisms is homeomorphism induced, then must all be homeomorphism induced? (**P 969**)

(2) Are all splitting homomorphism equivalences homeomorphism induced so long as the manifolds have homeomorphic boundaries? (**P 970**)

REFERENCES

[1] J. S. Birman, *Mapping class groups and their relationship to braid groups*, Communications on Pure and Applied Mathematics 22 (1969), p. 213-238.

[2] J. S. Downing, *Decomposing compact 3-manifolds into homeomorphic handlebodies*, Proceedings of the American Mathematical Society 24 (1970), p. 241-244.

[3] W. Jaco, *Heegaard splittings and splitting homomorphism*, Transactions of the American Mathematical Society 144 (1969), p. 365-380.

[4] C. D. Papakyriakopoulos, *On solid tori*, Proceedings of the London Mathematical Society 7 (1957), p. 281-299.

[5] — *On Dehn's lemma and the asphericity of knots*, Annals of Mathematics 66 (1967), p. 1-26.

[6] J. Stallings, *How not to prove the Poincaré conjecture* in *Topology Seminar Wisconsin*, Annals of Mathematical Studies 60, Princeton, New Jersey, 1965.

[7] H. Zieschang, *On simple systems of paths on complete pretzels*, American Mathematical Society Translations 92 (1970), p. 127-137.

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