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JOINT SPECTRA AND MULTIPLICATIVE FUNCTIONALS

BY

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In [1] it is proved that if the joint spectrum of an arbitrary n-tuple of elements in a Banach algebra A is non-empty, then A has a multiplicative linear functional. A similar theorem for the joint approximate point spectrum is obtained in [9]. In this paper we introduce and study a wide class of joint spectra for which a result of this type holds true.

Let A be a complex Banach algebra with the unit 1. We shall write $\mathscr{F}(A)$ for the family of all finite subsets of A. The usual spectrum of an elements $a \in A$ will be denoted by $\sigma(a)$, and its spectral radius by $\|a\|_s$. The symbol $\operatorname{rad} A$ will stand for the (Jacobson) radical of A, i.e., the intersection of the kernels of all irreducible representations of A. We shall denote by $\bar{\Omega}$ the closure (in the usual topology) of the subset Ω of C^n, while $\partial\Omega$ will stand for its boundary. We shall use polynomials over C with "non-commutative indeterminates" $x_1, \ldots, x_n$. Every such polynomial can be regarded as an element of the free associative algebra generated by the symbols $x_1, \ldots, x_n$ (cf. [2], [3] or [4]).

Now we introduce a class of joint spectra we are going to deal with in the paper.

DEFINITION 1. Let A be a unital, complex Banach algebra. A function $\tilde{\sigma}$ which assigns to each set $\{a_1, \ldots, a_n\}$ from $\mathscr{F}(A)$ a compact subset of C^n (possibly empty) is called a *generalized joint spectrum* on A if it satisfies the following three conditions:

(I) $$\tilde{\sigma}(a_1, \ldots, a_n) \subset \prod_{k=1}^{n} \sigma(a_k)$$

for every $\{a_1, \ldots, a_n\} \in \mathscr{F}(A)$. (For $\{a_1, \ldots, a_n\} \in \mathscr{F}(A)$ we shall write $\tilde{\sigma}(a_1, \ldots, a_n)$ instead of $\tilde{\sigma}(\{a_1, \ldots, a_n\})$.)

(II) $$p\tilde{\sigma}(a_1, \ldots, a_n) \subset \tilde{\sigma}(p(a_1, \ldots, a_n))$$

for all $\{a_1, \ldots, a_n\} \in \mathscr{F}(A)$ and an arbitrary m-tuple $p = (p_1, \ldots, p_m)$ of polynomials in n indeterminates.

(III) $$\tilde{\sigma}(a_1, \ldots, a_n) \neq \emptyset$$

for any pairwise commuting elements $a_1, \ldots, a_n$ in A.

Remarks 1. Axiom (I) implies, in particular, that $\tilde{\sigma}(a) \subset \sigma(a)$ for every a in A.

2. A property of the map $\tilde{\sigma}$ given by axiom (II) is called the *one-way spectral mapping property*. We say that $\tilde{\sigma}$ has the *spectral mapping property* if it satisfies (II) with the inclusion replaced by the equality.

3. Axiom (III) is introduced to exclude from our considerations the empty function, i.e., the function which takes only one value, the empty set. It also implies that $\tilde{\sigma}(a) \neq \emptyset$ for each $a \in A$.

4. It is possible to define a generalized joint spectrum not only on the finite subsets but on all the subsets of the algebra A (cf. [7] or [12]). Since this more general setting is not necessary for our purpose, we shall confine ourselves to the joint spectra defined only on the finite subsets of the algebra A.

5. In [12] Żelazko introduced an axiomatic theory of joint spectra but he considered only commuting families of elements in A. It should be noticed that a generalized joint spectrum is translation invariant, and thus, restricted to commuting subsets of A, is a spectroid in the terminology of [12]. On the other hand, if we take a subspectrum $\tilde{\sigma}$ on A ($\tilde{\sigma}$ is a function defined on commuting subsets of A, which satisfies (I), (III), and has the spectral mapping property; see [12]) and extend it to non-commuting elements $a_1, \ldots, a_n$ in A by the formula $\tilde{\sigma}(a_1, \ldots, a_n) = \emptyset$, then we get a generalized joint spectrum on A.

We shall list some examples of joint spectra which satisfy axioms (I)–(III).

EXAMPLES 1. The *left joint spectrum* of an n-tuple $(a_1, \ldots, a_n)$ of elements in A, denoted by $\sigma_l(a_1, \ldots, a_n)$, is defined to be the subset of C^n consisting of those $(\lambda_1, \ldots, \lambda_n)$ for which the system $(a_1 - \lambda_1, \ldots, a_n - \lambda_n)$ generates a proper left ideal of A. (Here, $a_j - \lambda_j$ stands for $a_j - \lambda_j 1$.)

2. The *right joint spectrum* $\sigma_r(a_1, \ldots, a_n)$ is defined in a similar manner.

3. The *joint spectrum*, called sometimes *Harte's spectrum*, is defined to be the union of the left and the right joint spectra:

$$\sigma(a_1, \ldots, a_n) = \sigma_l(a_1, \ldots, a_n) \cup \sigma_r(a_1, \ldots, a_n).$$

In [2] and [3] Harte proved that σ_l, σ_r, and σ satisfy properties (I)–(III).

4. The *left approximate point spectrum* of an n-tuple $(a_1, \ldots, a_n)$ of elements in A, denoted by $\tau_l(a_1, \ldots, a_n)$, is defined to be the subset of C^n

consisting of those $(\lambda_1, \ldots, \lambda_n)$ for which there exists a sequence (u_k) of elements in A such that $\|u_k\| = 1$ for all k and

$$\lim_k \|(a_j - \lambda_j) u_k\| = 0 \quad \text{for } j = 1, \ldots, n.$$

5. A definition of the *right approximate point spectrum* $\tau_r(a_1, \ldots, a_n)$ is similar.

6. The *joint approximate point spectrum* $\tau(a_1, \ldots, a_n)$ is the union of the left and the right approximate point spectra:

$$\tau(a_1, \ldots, a_n) = \tau_l(a_1, \ldots, a_n) \cup \tau_r(a_1, \ldots, a_n).$$

In [2] Harte proved that these approximate point spectra are compact subsets of C^n and satisfy conditions (I) and (II). Żelazko showed that they are non-empty for n-tuples consisting of pairwise commuting elements (see [11] and also [7], p. 134).

7. The *bicommutant spectrum* $\sigma''(a_1, \ldots, a_n)$ is defined to be the Harte spectrum of the n-tuple $(a_1, \ldots, a_n)$ in its bicommutant $\{a_1, \ldots, a_n\}''$.

In [4] Harte proved that the bicommutant spectrum satisfies axioms (I)–(III) of Definition 1.

Now we shall prove the previously mentioned result. We start with the following lemma which seems to be interesting in itself.

LEMMA 1. *Let $\tilde{\sigma}$ be a generalized joint spectrum on a Banach algebra A. If a function $\varphi\colon A \to C$ has the property*

$$(\varphi(a), \varphi(b)) \in \tilde{\sigma}(a, b)$$

for every a and b in A, then it is linear and multiplicative.

Proof. We need the Kowalski–Słodkowski theorem (see [6]) which reads as follows: If a function $\varphi\colon A \to C$ satisfies the following two conditions:

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(a) - \varphi(b) \in \sigma(a-b)$$

for arbitrary a and b in a Banach algebra A, then it is a multiplicative (linear) functional on A.

Thus we see that it is enough to show that the function φ satisfies the assumptions of this theorem. By (I) we get

$$(\varphi(0), \varphi(0)) \in \tilde{\sigma}(0, 0) \subset \sigma(0) \times \sigma(0) = \{(0, 0)\},$$

which implies $\varphi(0) = 0$. Further, by (II) and (I) we obtain

$$\varphi(a) - \varphi(b) \in \tilde{\sigma}(a-b) \subset \sigma(a-b)$$

for arbitrary a and b in A, and we are done.

PROPOSITION 1. *Let $\tilde{\sigma}$ be a generalized joint spectrum on a Banach* *algebra A. If*

$$\tilde{\sigma}(a_1, \ldots, a_n) \neq \emptyset \quad \textit{for every } \{a_1, \ldots, a_n\} \in \mathcal{F}(A),$$

then there exists a multiplicative functional φ on A. Moreover,

$$(\varphi(a_1), \ldots, \varphi(a_n)) \in \tilde{\sigma}(a_1, \ldots, a_n)$$

for an arbitrary $\{a_1, \ldots, a_n\} \in \mathcal{F}(A)$.

The author is indebted to Dr. K. Jarosz who supplied the following elegant and short proof.

Proof. We assume that $\tilde{\sigma}(a_1, \ldots, a_n)$ is always non-empty for finitely many elements $a_1, \ldots, a_n$ in A. Let

$$K = \prod_{a \in A} \sigma(a).$$

Then by the Tikhonov theorem it is a compact set (with respect to the product topology). Let us put

$$\omega(a_1, \ldots, a_n) = \{(\lambda_a)_{a \in A} \in K \colon (\lambda_{a_1}, \ldots, \lambda_{a_n}) \in \tilde{\sigma}(a_1, \ldots, a_n)\},$$

where $\{a_1, \ldots, a_n\} \in \mathcal{F}(A)$. This set is non-empty and compact, and moreover by (II) we get

$$\omega(a_1, \ldots, a_n, b_1, \ldots, b_m) \subset \omega(a_1, \ldots, a_n) \cap \omega(b_1, \ldots, b_m)$$

for arbitrary $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_m\}$ in $\mathcal{F}(A)$. Thus the family $\{\omega(a_1, \ldots, a_n)\}$, where $\{a_1, \ldots, a_n\}$ runs over all elements of $\mathcal{F}(A)$, has the finite intersection property. Therefore, its intersection is non-empty. If $(\lambda_a)_{a \in A}$ belongs to this intersection, then the function $\varphi \colon A \to C$ defined by the formula $\varphi(a) = \lambda_a$ is, in view of Lemma 1, a multiplicative functional on A. Moreover, it is obvious that

$$(\varphi(a_1), \ldots, \varphi(a_n)) \in \tilde{\sigma}(a_1, \ldots, a_n)$$

for each $\{a_1, \ldots, a_n\} \in \mathcal{F}(A)$.

Remark. The above result is a generalization of the main theorems in [1] and [9].

We shall write $\mathfrak{M}(A)$ for the family of all multiplicative functionals of the algebra A while $\mathfrak{M}(\tilde{\sigma}; A)$ will denote the set of all such functionals φ for which

$$(\varphi(a_1), \ldots, \varphi(a_n)) \in \tilde{\sigma}(a_1, \ldots, a_n)$$

for every $\{a_1, \ldots, a_n\} \in \mathcal{F}(A)$. Obviously, $\mathfrak{M}(\tilde{\sigma}; A) \subset \mathfrak{M}(A)$ and both of them can be empty.

For the examples of afore-named generalized joint spectra the following equalities hold true:

$$\mathfrak{M}(\sigma_l; A) = \mathfrak{M}(\sigma_r; A) = \mathfrak{M}(\sigma; A) = \mathfrak{M}(\sigma''; A) = \mathfrak{M}(A).$$

We have

$$\mathfrak{M}(\tau_l; A) = \{\varphi \in \mathfrak{M}(A)\colon \ker\varphi \text{ consists of joint left topological divisors of zero}\}.$$

(A subset S of a Banach algebra A consists of *joint left topological divisors of zero* if for every finite subset $\{x_1, \ldots, x_n\}$ of S there exists a sequence (z_k) of elements in A such that $\|z_k\| = 1$ for every k and $\lim_k \|x_j z_k\| = 0$ for $j = 1, \ldots, n$; cf. [9].) Similarly,

$$\mathfrak{M}(\tau_r; A) = \{\varphi \in \mathfrak{M}(A)\colon \ker\varphi \text{ consists of joint right topological divisors of zero}\}.$$

We have

$$\mathfrak{M}(\tau_l; A) \cup \mathfrak{M}(\tau_r; A) = \mathfrak{M}(\tau; A).$$

It is easy to give an example of a Banach algebra A for which $\mathfrak{M}(\tau; A) \neq \mathfrak{M}(A)$ (see [9]).

Now we shall recall a notion of the so-called "projection property" which plays an important role in the investigation of joint spectra. For each increasing sequence of indices

$$1 \leqslant j_1 < j_2 < \ldots < j_m \leqslant n$$

let us denote by $P_{j_1 \ldots j_m}$ the projection from C^n onto C^m defined by the formula

$$P_{j_1 \ldots j_m}(\lambda_1, \ldots, \lambda_n) = (\lambda_{j_1}, \ldots, \lambda_{j_m}).$$

If

$$P_{j_1 \ldots j_m} \tilde{\sigma}(a_1, \ldots, a_n) = \tilde{\sigma}(a_{j_1}, \ldots, a_{j_m})$$

for arbitrary $1 \leqslant j_1 < j_2 < \ldots < j_m \leqslant n$ and all $a_1, \ldots, a_n$ in A, then we say that the generalized joint spectrum $\tilde{\sigma}$ has the *projection property* on the algebra A.

We shall prove the following

PROPOSITION 2. *A generalized joint spectrum $\tilde{\sigma}$ on a Banach algebra A has the projection property if and only if*

$$(*) \qquad \tilde{\sigma}(a_1, \ldots, a_n) = \{(\varphi(a_1), \ldots, \varphi(a_n))\colon \varphi \in \mathfrak{M}(\tilde{\sigma}; A)\}$$

for every $\{a_1, \ldots, a_n\} \in \mathscr{F}(A)$.

Proof. We assume that $\tilde{\sigma}$ has the projection property. Then, obviously,

$$\tilde{\sigma}(a_1, \ldots, a_n) \neq \emptyset \quad \text{for every } \{a_1, \ldots, a_n\} \in \mathscr{F}(A).$$

Let an arbitrary n-tuple $(a_1, \ldots, a_n)$ of elements in A be fixed from now on. Taking $(\lambda_1, \ldots, \lambda_n) \in \tilde{\sigma}(a_1, \ldots, a_n)$, we have to show the existence of a multiplicative functional $\varphi \in \mathfrak{M}(\tilde{\sigma}; A)$ such that $\varphi(a_j) = \lambda_j$ for $j = 1, \ldots, n$. Let, as before,

$$K = \prod_{a \in A} \sigma(a).$$

For an arbitrary m-tuple $(b_1, \ldots, b_m)$ of elements in A we write

$$\delta(b_1, \ldots, b_m) = \{(\mu_a)_{a\in A} \in K\colon \mu_{a_j} = \lambda_j \quad \text{for } j = 1, \ldots, n \text{ and } (\mu_{b_1}, \ldots, \mu_{b_m}) \in \tilde{\sigma}(b_1, \ldots, b_m)\}.$$

The projection property of $\tilde{\sigma}$ implies that $\delta(b_1, \ldots, b_m)$ is always non-empty. It is obvious that it is compact and, moreover,

$$\delta(b_1, \ldots, b_m, c_1, \ldots, c_p) \subset \delta(b_1, \ldots, b_m) \cap \delta(c_1, \ldots, c_p)$$

for arbitrary $\{b_1, \ldots, b_m\}$ and $\{c_1, \ldots, c_p\}$. Thus the family $\{\delta(b_1, \ldots, b_m)\}$, where $\{b_1, \ldots, b_m\}$ runs over $\mathscr{F}(A)$, has the finite intersection property. Taking a point $(\mu_a)_{a\in A}$ from the intersection of this family we get the function $\varphi\colon A \to \boldsymbol{C}$, $\varphi(a) = \mu_a$, which, in view of Lemma 1, is a multiplicative functional on A and has the required property.

It is evident that if for every finite set $\{a_1, \ldots, a_n\}$ we have (*), then $\tilde{\sigma}$ has the projection property on the algebra A.

As an immediate consequence of Proposition 2 we get the following

COROLLARY 1. *A generalized joint spectrum $\tilde{\sigma}$ has the projection property on a Banach algebra A if and only if it has the spectral mapping property on this algebra.*

Remark. In [5] (see also [7], p. 146) it is shown that a semispectrum $\tilde{\sigma}$ ($\tilde{\sigma}$ is a function defined on commuting subsets of A which has the projection property, is always non-empty, and $\tilde{\sigma}(a_1+\lambda_1, \ldots, a_n+\lambda_n) = (\lambda_1, \ldots, \lambda_n) + \tilde{\sigma}(a_1, \ldots, a_n)$; see [12]) satisfies the one-way spectral mapping property if and only if it has the spectral mapping property. Hence Corollary 1 can be regarded as a counterpart of this result in the non-commutative case.

Let us introduce one more axiom:

(IV) $$\max\{|\lambda|\colon \lambda \in \tilde{\sigma}(a)\} = \|a\|_s \quad \text{for all } a \in A.$$

COROLLARY 2. *If a generalized joint spectrum $\tilde{\sigma}$ satisfies* (IV) *and has the projection property on a Banach algebra A, then*

$$\bigcap\{\ker\varphi\colon \varphi \in \mathfrak{M}(\tilde{\sigma}; A)\} = \operatorname{rad} A.$$

In particular, the algebra $A/\operatorname{rad} A$ is commutative.

Proof. Let us take arbitrary

$$a \in \bigcap \{\ker \varphi \colon \varphi \in \mathfrak{M}(\tilde{\sigma}; A)\} \quad \text{and} \quad b \in A.$$

Then

$$\|ab\|_s = \max\{|\lambda| \colon \lambda \in \tilde{\sigma}(ab)\} = \max\{|\varphi(ab)| \colon \varphi \in \mathfrak{M}(\tilde{\sigma}; A)\}$$
$$= \max\{|\varphi(a)\varphi(b)| \colon \varphi \in \mathfrak{M}(\tilde{\sigma}; A)\} = 0,$$

and therefore $a \in \operatorname{rad} A$. Thus

$$\bigcap \{\ker \varphi \colon \varphi \in \mathfrak{M}(\tilde{\sigma}; A)\} \subset \operatorname{rad} A.$$

The converse inclusion is obvious by the definition of the radical.

Remarks. 1. All the examples of generalized joint spectra afore-cited satisfy axiom (IV) since for such a spectrum $\tilde{\sigma}$ we have $\partial\sigma(a) \subset \tilde{\sigma}(a)$ for all $a \in A$.

2. The commutativity of the algebra modulo its radical suffices for the joint spectra σ_l, σ_r, and σ to have the projection property (see [1]) whereas this is not a sufficient condition for the approximate point spectra τ_l, τ_r, and τ to have this property (see [9]). We shall also show that the bicommutant spectrum σ'' need not have the projection property if $A/\operatorname{rad} A$ is commutative. To see this we shall take the modification of the famous Taylor example (see [10], and also [7]) given by Harte in [4], pp. 300–302. Let

$$\Delta = \{z \in \mathbf{C} \colon |z| < 1\},$$
$$U = 3\Delta \times 3\Delta, \qquad W = \Delta \times \Delta, \qquad \text{and} \qquad V = \overline{U \setminus W}.$$

Symbols $\mathscr{C}(V)$, $\mathscr{C}^{(1)}(V)$, $\mathscr{A}(V)$ will denote the algebras of all continuous functions on V, all continuously differentiable functions on V, and all continuous functions on V which are analytic in its interior, respectively. Let A be the algebra of all (bounded and linear) operators on $X = \mathscr{C}(V) \oplus \mathscr{C}^{(1)}(V)$ of the form

$$(f, g) \mapsto (hf + Kg, hg),$$

where $h \in \mathscr{C}^{(1)}(V)$ and K is an arbitrary operator from $\mathscr{C}^{(1)}(V)$ into $\mathscr{C}(V)$. It is convenient to represent elements of A in the matrix form

$$\begin{bmatrix} L_h & K \\ 0 & L_h \end{bmatrix},$$

where L_h denotes the multiplication operator by the function h. Then it is easy to see that the radical of A consists of the operators $(f, g) \mapsto (Kg, 0)$, and so the algebra A is commutative modulo its radical. Moreover, multiplicative functionals on A are given by the evaluation of the function h at points of V, i.e., $\varphi_z \in \mathfrak{M}(A)$ if and only if it takes the form

$$\varphi_z\left(\begin{bmatrix} L_h & K \\ 0 & L_h \end{bmatrix}\right) = h(z) \quad (z \in V).$$

Now we take operators a_j given by $a_j(f, g) = (z_j f, z_j g)$, $j = 1, 2$, and $z = (z_1, z_2)$. Then reasoning as in [4] we get the following inclusions for the bicommutant of $\{a_1, a_2\}$ in the algebra A:

$$D_0 = \left\{\begin{bmatrix} L_h & 0 \\ 0 & L_h \end{bmatrix}:\ h \in \mathscr{A}(V)\right\} \subset \{a_1, a_2\}''$$

$$\subset D_1 = \left\{\begin{bmatrix} L_h & K \\ 0 & L_h \end{bmatrix}:\ h \in \mathscr{A}(V),\ KL_{z_j} = L_{z_j}K \text{ for } j = 1, 2\right\}.$$

The algebras D_0 and D_1 have more multiplicative functionals than A since every function analytic in the interior of V extends analytically to U. Thus we have

$$\mathfrak{M}(D_0) = \mathfrak{M}(D_1) = \mathfrak{M}(\{a_1, a_2\}'') = \{\varphi_z\colon z \in \bar{U}\},$$

which implies that

$$\sigma''(a_1, a_2) = \bar{U} \neq V = \{(\varphi(a_1), \varphi(a_2))\colon \varphi \in \mathfrak{M}(A)\}.$$

Hence, in view of Proposition 2, the bicommutant spectrum does not have the projection property on A.

Now we shall define a partial order in the set $\Sigma(A)$ of all generalized joint spectra on a Banach algebra A and we shall prove that every Banach algebra has the largest generalized joint spectrum. We shall also show that there exist minimal generalized joint spectra.

Definition 2. For two generalized joint spectra $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ defined on a Banach algebra A we shall write $\tilde{\sigma}_1 \leqslant \tilde{\sigma}_2$ if

$$\tilde{\sigma}_1(a_1, \ldots, a_n) \subset \tilde{\sigma}_2(a_1, \ldots, a_n) \quad \text{for each } \{a_1, \ldots, a_n\} \in \mathscr{F}(A).$$

This is a *partial order* in the set $\Sigma(A)$.

We need the following lemma (cf. [12]):

Lemma 2. *If p is a continuous map from C^n into C^m and Ω is a relatively compact subset of C^n, then $p(\bar{\Omega}) = \overline{p(\Omega)}$.*

The proof of this lemma is straightforward, and therefore will be omitted here.

Proposition 3. *There exists the largest generalized joint spectrum $\sigma_{\max}$ on A.*

Proof. We shall proceed exactly in the same way as in the proof of Theorem 6.3 in [12]. Namely, for a fixed $\{a_1, \ldots, a_n\}$ from $\mathscr{F}(A)$ let us define

$$\sigma_{\max}(a_1, \ldots, a_n) = \overline{\bigcup \{\tilde{\sigma}(a_1, \ldots, a_n) \colon \tilde{\sigma} \in \Sigma(A)\}}.$$

It is clear that $\sigma_{\max}$ satisfies axioms (I) and (III) (and also (IV)). To see that axiom (II) is also satisfied let us take an arbitrary system $p = (p_1, \ldots, p_m)$ of polynomials in n variables and apply Lemma 2. Then we get

$$\begin{aligned} p\sigma_{\max}(a_1, \ldots, a_n) &= p\left(\overline{\bigcup \{\tilde{\sigma}(a_1, \ldots, a_n) \colon \tilde{\sigma} \in \Sigma(A)\}}\right) \\ &= \overline{p\left(\bigcup \{\tilde{\sigma}(a_1, \ldots, a_n) \colon \tilde{\sigma} \in \Sigma(A)\}\right)} \\ &= \overline{\bigcup \{p\tilde{\sigma}(a_1, \ldots, a_n) \colon \tilde{\sigma} \in \Sigma(a)\}} \\ &\subset \overline{\bigcup \left\{\tilde{\sigma}(p(a_1, \ldots, a_n)) \colon \tilde{\sigma} \in \Sigma(A)\right\}} = \sigma_{\max}(p(a_1, \ldots, a_n)). \end{aligned}$$

PROPOSITION 4. *$\Sigma(A)$ contains minimal elements.*

Proof. If $\{\tilde{\sigma}_\alpha\}$ is a totally ordered subfamily of $\Sigma(A)$, then, evidently,

$$\tilde{\sigma}_0 = \bigcap_\alpha \tilde{\sigma}_\alpha$$

satisfies axioms (I) and (II), and the finite intersection property implies that $\tilde{\sigma}_0$ has also property (III). Hence $\tilde{\sigma}_0$ is a generalized joint spectrum on A such that $\tilde{\sigma}_0 \leqslant \tilde{\sigma}_\alpha$ for all α. By the Kuratowski–Zorn lemma we obtain minimal elements in $\Sigma(A)$.

Remarks. 1. If an algebra A is commutative and φ is a fixed multiplicative functional on A, then the joint spectrum $\tilde{\sigma}_\varphi$ defined by the formula

$$\tilde{\sigma}_\varphi(a_1, \ldots, a_n) = \left\{(\varphi(a_1), \ldots, \varphi(a_n))\right\}$$

is a minimal generalized joint spectrum on this algebra.

2. If a Banach algebra A has a single point subspectrum $\tilde{\sigma}$ (i.e., the value of $\tilde{\sigma}$ always consists of a single point; see [8]), then the trivial extension of $\tilde{\sigma}$ to $\mathscr{F}(A)$ ($\tilde{\sigma}(a_1, \ldots, a_n) = \emptyset$ for non-commuting elements $a_1, \ldots, a_n$ in A) is a minimal generalized joint spectrum on A. For example, such a situation occurs when $A = B(J)$ or $A = B(J \oplus J)$, where J is the classical James space (see [8]).

3. In an arbitrary Banach algebra A the following relations hold true:

$$\tau_l \leqslant \sigma_l \leqslant \sigma \leqslant \sigma'', \quad \tau_r \leqslant \sigma_r \leqslant \sigma \leqslant \sigma'', \quad \tau \leqslant \sigma \leqslant \sigma''.$$

Moreover, in the axiomatic theory of Żelazko (see [12]) every joint spectrum (even subspectrum) must be contained in the bicommutant spectrum (which is neither a joint spectrum nor a subspectrum in this theory). These facts may suggest that the bicommutant spectrum is a good candidate for the largest generalized joint spectrum. Hence we ask the following question:

Do the largest generalized joint spectrum and the bicommutant spectrum coincide? **(P 1360)**(1)

If the answer to this question is negative we may still ask the following:

Does there exist a simple characterization of the largest generalized joint spectrum?

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(1) See *Problèmes*, p. 393.