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A NEW PROOF OF THE L^p-CONJECTURE FOR LOCALLY COMPACT ABELIAN GROUPS

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The L^p-conjecture for an arbitrary locally compact group G takes the following form: if μ is a left Haar measure on G, and $L^p(G) = L^p(G, \mu)$ is closed under convolution for some p, $1 < p < \infty$, then G is compact. The conjecture is completely solved (in the affirmative) for $2 \leqslant p < \infty$ and, if G is amenable or almost connected, for $1 < p < 2$. For abelian G, the first case settled, the result is due to Żelazko [7]; subsequent proofs were given in [6] and [5]. The purpose of this note is to give a new proof of the L^p-conjecture for locally compact abelian groups, which is, in some sense, the most natural possible in that G is shown to be compact by realizing it as a closed subgroup of a larger compact (Hausdorff) topological group.

THEOREM. *Let G be a locally compact abelian group, and let μ be a left ($=$ right) Haar measure on G. If $L^p(G) = L^p(G, \mu)$ is closed under convolution for some p, $1 < p < \infty$, then G is compact.*

Proof. Assume, without loss of generality ([4], Theorem 1.3), that the Haar measure μ has been adjusted so that $A = L^p(G)$ is a Banach algebra under convolution. Then A is a commutative (since G is abelian) semisimple Banach algebra ([4], Lemma 2.2). Let $M(A)$ be the commutative Banach algebra of all multipliers of A, and let $G(A)$ be the (abstract) group of all isometric onto multipliers of A (see [1] for definitions). The multiplier algebra $M(A)$ is contained in $B(A)$, the space of all bounded linear operators on A, and (since A is reflexive) the unit ball $B(A)_1$ of $B(A)$ is compact in the weak operator topology τ_{wo} ([2], Exercise 6, p. 512). It is easily verified that $M(A)$ is τ_{wo}-closed in $B(A)$; hence, its unit ball $M(A)_1$ is τ_{wo}-compact. Consequently, the group $G(A)$, equipped with the τ_{wo}-topology, is an abelian compact Hausdorff topological group ([1], Theorem 3). Now, each x in G defines an element T_x of $G(A)$ by $(T_x f)(y) = f(x^{-1}y)$, where $f \in L^p(G)$, $y \in G$, and the map $x \to T_x$ is a continuous group isomorphism from G into $G(A)$. In fact, a straightforward argument shows that $x \to T_x$ is a homeomorphism of G into $G(A) = (G(A), \tau_{\mathrm{wo}})$. Thus, G is realized as a topological subgroup of the compact topological group $G(A)$ and, since G is locally compact, it follows ([3], Theorem 5.11, p. 35) that G is closed in $G(A)$.

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