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A NOTE ON TOPOLOGICAL m-SPACES

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The purpose of this paper is to characterize 1-dimensional Peano continua admitting a mean. It is also shown that compact connected absolute neighborhood retracts (ANR's) that are imbeddable in R^3 admit a mean if and only if they are absolute retracts (AR's) whereas locally arcwise connected and semi-locally simply connected continua with a mean have finite fundamental groups.

A space X is said to *admit an n-mean* if there is a map (continuous function) $\mu: X^n \to X$, where $X^n = X \times \ldots \times X$ is the n-fold Cartesian product of X, satisfying the conditions

1. $\mu(x, \ldots, x) = x$ and

2. $\mu(x_1, x_2, \ldots, x_n) = \mu(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for every permutation σ of the set $\{1, 2, \ldots, n\}$.

Spaces with a mean are also referred to as *m-spaces*. Since, for $n = 1$, every space becomes an m-space, μ being the identity, we assume $n \geqslant 2$.

Definition. A map $f: X \to Y$ is an *r-map* if there is a map $g: Y \to X$ such that $f \circ g: Y \to Y$ is the identity on Y, i.e. g is a right inverse of f.

PROPOSITION 1. *If $f: X \to Y$ is an r-map and X admits a mean, then Y admits a mean.*

Proof. Let μ be an n-mean on X. The map

$$f \circ \mu \circ g^n: Y^n \to Y, \quad \text{where } g^n(y_1, \ldots, y_n) = \big(g(y_1), \ldots, g(y_n)\big)$$

is easily shown to be an n-mean on Y.

COROLLARY 2. *If Y is a retract of X and X is an m-space, then Y is an m-space.*

Proof. Let $r: X \to Y$ be a retraction. The identity map $i: Y \to Y$ is a right inverse of r, therefore Y admits a mean by Proposition 1.

Definition. A continuum (compact, connected) X is said to be *unicoherent* if, for any two subcontinua A and B such that $X = A \cup B$, $A \cap B$ is a continuum.

THEOREM 3. *For a 1-dimensional Peano continuum X the following are equivalent:*

(1) *X admits a mean,*

(2) *X is unicoherent,*

(3) *X is a dendrite.*

Proof. (1) $\Rightarrow$ (2). Suppose X admits a mean and X is not unicoherent. By a theorem of Borsuk ([3], Théorème 30, p. 184) X retracts to a simple closed curve contained in X. But a simple closed curve cannot admit a mean [1]. By Corollary 2, X cannot admit a mean. This contradiction establishes the implication.

(2) $\Rightarrow$ (3). Since X is 1-dimensional and unicoherent, it cannot contain a simple closed curve. Thus X is a locally connected continuum with only degenerate cyclic elements, therefore X is a dendrite ([12], 1.2 (i), p. 89).

(3) $\Rightarrow$ (1). One characterization of a dendrite X is that between any two points there is a unique arc ([12], 1.2 (ii), p. 89). If x and y are any two points of X and $\mu(x, y)$ is defined to be the unique mid-point of the arc $\widehat{xy}$, then, clearly, μ is a mean on X.

Remark. It follows from Theorem 3 that a 1-dimensional Peano continuum admits a mean if and only if it is unicoherent.

In fact, for continua that admit a mean, unicoherence is always implied ([10], Corollary 3).

Before proving the next theorem we list some pertinent facts:

FACT 1. *If X is a compact subset of a Euclidean space R^n and X admits a mean, then $R^n - X$ is connected* ([10], Corollary 2).

FACT 2. *All singular homology and cohomology groups of m-spaces cannot contain Z (the group of integers) as a summand* ([4], Section 5, p. 337).

FACT 3. *The fundamental group of an m-space is Abelian* ([4], Satz 5, p. 335) *and Z cannot occur as a summand.*

THEOREM 4. *A compact connected ANR which is imbeddable in R^3 admits a mean if and only if it is an AR.*

Proof. Let X be a compact connected ANR subset of R^3 and let X admit a mean. To show that X is an AR it will suffice to show that X is simply connected and that all its Čech homology groups over the coefficient group Z of integers are trivial by [2], Theorem 10.8, p. 124. Since for ANR's the Čech and singular homology (cohomology) groups are equivalent [8], the homology and cohomology groups employed below are singular.

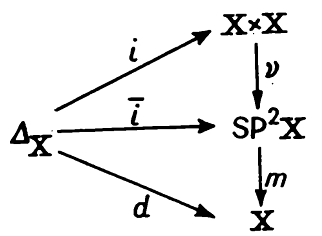
We first note that since X is an ANR, X is the r-image of a polyhedron ([2], Theorem 10.1, p. 122) and as such it has finitely generated homology and cohomology groups. From dimension consideration ($\dim X \leqslant 3$), $H^k(X) = 0$ for $k \geqslant 3$ ([7], Theorem VIII.4, p. 152). Since X admits a mean, X is unicoherent by the Remark above, and $H^1(X) = 0$ ([8], Theorem 2).

By Alexander duality ([11], Theorem 16, p. 296),

$$\tilde{H}_0(R^3 - X) = H^2(X),$$

where $\tilde{H}_0(R^3 - X)$ is the reduced group in dimension zero. It follows from Fact 1 that $\tilde{H}_0(R^3 - X) = 0$ since $R^3 - X$ is connected, and hence $H^2(X) = 0$. Since, for spaces with finitely generated homology groups $H_q(X)$ for all q, the torsion submodule of $H^q(X)$ is equal the torsion submodule of $H_{q-1}(X)$ ([11], Corollary 4, p. 244), we conclude, by Fact 2, that $H_q(X) = 0$ for $q \geqslant 1$. Finally, by Fact 3, $\pi(X) \approx H_1(X) = 0$. Thus X is simply connected with all its homology groups zero and, therefore, X is an AR.

For the converse, let X be a compact AR subset of R^3 and let $\Delta_X = \{(x, x) \colon x \in X\}$. Denote by $\mathrm{SP}^2 X$ the quotient space $X \times X/\sim$, where $\sim$ is the equivalence relation defined by $(x, y) \sim (y, x)$ for $x, y \in X$. If $d : \Delta_X \to X$ and $\bar{i} : \Delta_X \to \mathrm{SP}^2 X$ are defined by $d(x, x) = x$ and $\bar{i}(x, x) = \overline{(x, x)}$, respectively, then the following diagram commutes:



Here i is the inclusion map, ν the quotient map, and $m\bar{i} = d$, i.e. m is defined on $\bar{i}(\Delta_X)$ which is a closed subset of $\mathrm{SP}^2 X$. The quotient space $\mathrm{SP}^2 X$ is metrizable and, since X is an AR, m can be extended over $\mathrm{SP}^2 X$ ([2], Theorem 4.2, p. 87). The composition $m \circ \nu$ is a 2-mean on X since

1. $m\nu(x, x) = m\overline{(x, x)} = x$, and
2. $m\nu(x, y) = m\nu(y, x)$ in view of $\nu(x, y) = \nu(y, x)$.

Definition. A space X is said to be *semi-locally simply connected* if every point $x \in X$ is contained in a neighborhood U such that $\pi(U, x) \to \pi(X, x)$ is the trivial homomorphism.

THEOREM 5. *Let X be a locally arcwise connected and semi-locally simply connected continuum that admits a mean. Then $\pi(X, x_0)$ is finite.*

Proof. It is known [9] that if a space is compact, connected, locally arcwise connected and semi-locally simply connected, then its fundamental group is finitely generated. For such a space X, $\pi(X, x_0)$ is finitely generated Abelian group by Fact 3, since X is an m-space. By the decomposition theorem for finitely generated Abelian groups and by Fact 3, $\pi(X, x_0)$ is the direct sum of finitely many cyclic groups none of which is Z. Therefore, $\pi(X, x_0)$ is finite.

COROLLARY 6. *Let X be as in Theorem 5 and let k be the order of $\pi(X, x_0)$. If n is a positive integer such that $(k, n) \neq 1$, i.e. k and n are relatively prime, then X does not admit an n-mean.*

Proof. By a *homomorphic n-mean* on an Abelian group G we understand a function $m\colon G^n \to G$ such that

1. $m(g, g, \ldots, g) = g, \ g \in G$,
2. $m(g_1, g_2, \ldots, g_n) = m(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$, and
3. $m(g_1+g_1', \ldots, g_n+g_n') = m(g_1, \ldots, g_n)+m(g_1', \ldots, g_n')$.

If X admits an n-mean, then $\pi(X, x_0)$ admits a homomorphic n-mean and the homomorphism $a \mapsto na$, $a \in \pi(X, x_0)$, is an automorphism ([4], Satz 3). By the hypothesis, $(k, n) = d \neq 1$. If p is a prime divisor of d, then p is also a prime divisor of both k and n, and $n = n_0 p$ for some positive integer n_0. By Cauchy's theorem ([6], Theorem 2.2.5, p. 74), there exists an element $\beta \in \pi(X, x_0)$ such that $\beta \neq e$ and $p\beta = e$. Thus $\beta \mapsto n\beta = n_0(p\beta) = n_0 e = e$, and hence the homomorphism $a \mapsto na$ is not an automorphism since its kernel contains $\beta \neq e$. Therefore, $\pi(X, x_0)$ is not an m-group (a group is an *m-group* if it admits a homomorphic n-mean) and, consequently, X does not admit an m-mean.

QUESTIONS. 1. Is a locally connected continuum in R^3 that admits a mean contractible? (**P 924**)

2. If a space X, as in Corollary 6, admits a mean, does it imply that $\pi(X, x) = 0$? (**P 925**)

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