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ON THE THEOREM OF F. AND M. RIESZ

BY

LOUIS PIGNO AND SADAHIRO SAEKI (MANHATTAN, KANSAS)

It is a well-known fact that the F. and M. Riesz theorem and the Rudin–Carleson theorem are equivalent (cf. [2] and [5]). In this note we shall give another condition which is equivalent to these two results. We shall also give a proof of the F. and M. Riesz theorem in the logmodular algebra setting. For an elegant proof of this classical result for the circle, we refer to R. Doss' paper [4].

Let G be a compact abelian group with dual Γ, and let m_G be the normalized Haar measure on G [9]. The Fourier transform of a measure $\mu \in M(G)$ is defined by

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma}\, d\mu \qquad (\gamma \in \Gamma).$$

A subset E of Γ is called a *Riesz set* if $M_E(G) \subset M_a(G)$, where $M_a(G) = \{ \mu \in M(G) : \mu \ll m_G \}$ and

$$M_E(G) = \{ \mu \in M(G) : \hat{\mu} = 0 \text{ off } E \}.$$

Now let $F \subset \Gamma$, and let $C_F(G)$ denote the closed linear span of F in $C(G)$. We shall call F a *CR-set* if F satisfies the following condition: whenever K is a compact subset of G with $m_G(K) = 0$ and $\phi : G \to (0,1]$ is a lower-semicontinuous function with $\phi = 1$ on K, then each $f \in C(K)$ extends to a function $\tilde{f} \in C_F(G)$ in such a way that $|\tilde{f}| \le \|f\|_K \phi$ on G, where $\|f\|_K$ denotes the uniform norm of f over K.

THEOREM 1. *Let G be a compact abelian group with dual Γ, and let E be a subset of Γ. Then the following assertions are equivalent*:

(a) *E is a Riesz set.*

(b) *$F = \Gamma \setminus E$ is a CR-set.*

(c) *There exists a finite positive constant B with the following property: given a compact subset K of G of Haar measure zero, $f \in C(K)$ and $\varepsilon > 0$, there exists $g \in C_F(G)$ such that*

$$\|g - f\|_K < \varepsilon \quad \text{and} \quad \|g\|_G \le B\|f\|_K.$$

The equivalence of (a) and (b) is essentially due to E. Bishop [2]; see also [5]. That (b) implies (c) is banal. To prove that (c) implies (a), we need a lemma.

LEMMA 2. *Given $\nu \in M(G)$, $\varepsilon > 0$, and a finite set $F_0 \subset \Gamma$, there exist $c_1, \ldots, c_n \in \mathbf{C}$ and nonempty open sets $U_1, \ldots, U_n \subset G$ such that*

(i) $|c_1| + \ldots + |c_n| \leq \|\nu\|$, *and*

(ii) $|\hat{\nu}(\gamma) - \sum_{k=1}^{n} c_k \overline{\gamma}(x_k)| < \varepsilon$ *for all $\gamma \in F_0$ and all $x_k \in U_k$ $(1 \leq k \leq n)$.*

If ν is a probability measure, such c_k can be chosen to satisfy

(iii) $c_k \geq 0$ $\forall k$, *and* $c_1 + \ldots + c_n = 1$.

Proof. Let $V = \{x \in G : |\gamma(x) - 1| < \varepsilon/2 \ \forall \gamma \in F_0\}$, so that V is a neighborhood of 0 in G. Since G is compact, we can find finitely many elements $y_1, \ldots, y_n$ of G such that $G = \bigcup_{k=1}^{n} U_k$, where $U_k = y_k + V$ $(1 \leq k \leq n)$. Notice that $x, y \in U_k$ for some k implies that $|\gamma(x) - \gamma(y)| < \varepsilon$ for all $\gamma \in F_0$.

Now let $E_k = U_k \setminus (\bigcup_{j=1}^{k-1} U_j)$ and $c_k = \nu(E_k)$ for $1 \leq k \leq n$. Then (i) is obvious. If $\gamma \in F_0$ and $x_k \in U_k$ for all k, then

$$\left|\hat{\nu}(\gamma) - \sum_{k=1}^{n} c_k \overline{\gamma}(x_k)\right| \leq \sum_{k=1}^{n} \left| \int_{E_k} \{\overline{\gamma} - \overline{\gamma}(x_k)\}\, d\nu \right| \leq \sum_{k=1}^{n} \varepsilon |\nu|(E_k) = \varepsilon \|\nu\|,$$

which completes the proof of our lemma.

Proof that (c) implies (a). Without loss of generality, assume that E contains $1 \in \Gamma$. Given $\mu \in M_E(G)$, let $\mu = \mu_a + \mu_s$ denote the Lebesgue decomposition of μ with respect to m_G. We must prove that $\mu_s = 0$. Notice that

$$\int \overline{f}\, d\mu_s = -\int \overline{f}\, d\mu_a \quad \forall f \in C_F(G). \tag{1}$$

Given $\varepsilon > 0$, choose a finite set $F_0 \subset \Gamma$ so that

$$\left| \int \overline{f}\, d\mu_a \right| \leq \varepsilon \|f\|_G \tag{2}$$

for all $f \in C(G)$ with $\hat{f} = 0$ on F_0 (cf. [3]). We apply Lemma 2 to $\nu = m_G$ to obtain $c_1, \ldots, c_n \geq 0$ and nonempty open subsets $U_1, \ldots, U_n$ of G such that

$$c_1 + \ldots + c_n = 1 \quad \text{and} \tag{3}$$

$$\left| \sum_{k=1}^{n} c_k \gamma(x_k) \right| < \varepsilon / (B \cdot |F_0|) \quad \forall \gamma \in F_0 \setminus \{1\} \ \ \& \ \ x_k \in U_k, \tag{4}$$

where $|F_0|$ denotes the cardinality of F_0.

Now we claim that there exist $x_k \in U_k$ $(1 \leq k \leq n)$ such that

$$\delta_{x_k} * \mu_s \perp \delta_{x_j} * \mu_s \quad \forall k \neq j, \tag{5}$$

where δ_x denotes the unit point mass at x. Indeed, pick any $x_1 \in U_1$. Suppose that $x_1 \in U_1, \ldots, x_m \in U_m$ have been chosen for some $m < n$. Since μ_s is a singular measure, it is easy to find $x_{m+1} \in U_{m+1}$ so that (5) holds for all $k \leq m$ and $j = m+1$. This completes our induction, thereby confirming our claim.

Now define $\nu = \sum_{k=1}^{n} c_k \delta_{x_k} * \mu_s$. Then ν is singular and $\|\nu\| = \|\mu_s\|$ by (3) and (5). It follows from (c) and the regularity of ν that there exists $g \in C_F(G)$ such that

$$(6) \qquad \|g\|_G \leq B \quad \text{and} \quad \int \overline{g}\, d\nu > \|\mu_s\| - \varepsilon.$$

Define $h \in C(G)$ by setting

$$(7) \qquad h(y) = \sum_{k=1}^{n} c_k g(y + x_k) \quad \forall y \in G.$$

Then $\|h\|_G \leq B$ by (3) and (6), and

$$(8) \qquad \int \overline{h}\, d\mu_s = \sum_{k=1}^{n} c_k \int \overline{g}(y + x_k)\, d\mu_s(y) = \int \overline{g}\, d\nu > \|\mu_s\| - \varepsilon$$

by (7), the definition of ν and (6). Moreover, we have

$$(9) \qquad h \in C_F(G) \quad \text{and} \quad |\hat{h}(\gamma)| < \varepsilon/|F_0| \quad \forall \gamma \in F_0.$$

In fact, $\gamma \in \Gamma$ implies

$$\hat{h}(\gamma) = \sum_{k=1}^{n} c_k \int \overline{\gamma}(y) g(y + x_k)\, dy = \Big\{ \sum_{k=1}^{n} c_k \gamma(x_k) \Big\} \cdot \hat{g}(\gamma)$$

by (7). Since $g \in C_F(G)$ and $1 \in E$, it follows that $h \in C_F(G)$ and $\hat{h}(1) = 0$. If $1 \neq \gamma \in F_0$, then $|\hat{h}(\gamma)| \leq \varepsilon |\hat{g}(\gamma)|/(B|F_0|) \leq \varepsilon/|F_0|$ by (4) and (6). All these together confirm (9).

Finally, set $f = h - \sum\{ \hat{h}(\gamma)\gamma : \gamma \in F_0 \}$, so that $\hat{f} = 0$ on F_0. Moreover, $f \in C_F(G)$ and $\|f - h\|_G < \varepsilon$ by (9). In particular, $\|f\|_G < \|h\|_G + \varepsilon \leq B + \varepsilon$. Accordingly

$$\begin{aligned} \|\mu_s\| - \varepsilon &< \int \overline{h}\, d\mu_s && \text{by (8)} \\ &\leq \Big| \int \overline{f}\, d\mu_s \Big| + \|f - h\|_G \|\mu_s\| \\ &\leq \Big| \int \overline{f}\, d\mu_a \Big| + \varepsilon \|\mu\| && \text{by (1)} \\ &\leq \varepsilon(B + \varepsilon) + \varepsilon \|\mu\| && \text{by (2).} \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\mu_s = 0$, as desired.

Remark. We feel that all the conditions in Theorem 1 are equivalent to the following one:

(d) $C_F(G) \mid K = C(K)$ $\forall$ compact $K \subset G$ with $m_G(K) = 0$.

If (d) is the case and if K is as above, then an easy application of the closed graph theorem yields a finite constant B_K such that each $f \in C(K)$ has an extension $g \in C_F(G)$ such that $\|g\|_G \le B_K\|f\|_K$. Thus the problem is as to whether we can choose such B_K independently of K.

Now let X be a compact Hausdorff space, and let A be a logmodular algebra on X. The latter means that A is a uniformly closed subalgebra of $C(X)$ such that A contains the constants and such that $\{\log|f| : f \in A^{-1}\}$ is uniformly dense in $C_{\mathbf{R}}(X)$, where A^{-1} denotes the set of all invertible elements of the Banach algebra A (cf. [7], [8]). We choose and fix any probability measure $m \in M(X)$ which is multiplicative on A. Thus

$$A_0 = \left\{ f \in A : \int f\,dm = 0 \right\}$$

is a closed maximal ideal in A.

THEOREM 3. *Given a compact subset K of X with $m(K) = 0$ and $\varepsilon > 0$, there exists $g \in A$ such that $|g| < 1$ on X, $|g| < \varepsilon$ on K, and*

$$\int |g - 1|\,dm < \varepsilon.$$

Proof (cf. Lemma 3 of [10]). We may assume that $\varepsilon < 1$. Since A is logmodular and $m(K) = 0$, the regularity of m ensures that there exists $g \in A^{-1}$ such that $|g| < 1$ on X, $|g| < \varepsilon$ on K, and

$$\int \log|g|\,dm > \log(1 - 2^{-1}\varepsilon^2). \tag{10}$$

Notice that m is an Arens–Singer measure ([1], [8]); hence

$$\int \log|g|\,dm = \log\left|\int g\,dm\right|. \tag{11}$$

Replacing g by cg for some complex number c of absolute value one, we may assume that $\int g\,dm > 0$. Then $\int g\,dm > 1 - 2^{-1}\varepsilon^2$ by (10) and (11), and so

$$\int |g - 1|^2\,dm = \int (|g|^2 - 2\mathrm{Re}\,g + 1)\,dm < 1 - 2(1 - 2^{-1}\varepsilon^2) + 1 = \varepsilon^2.$$

Since m is a probability measure, it follows from Schwarz' inequality that $\|g - 1\|_1 \le \|g - 1\|_2 < \varepsilon$.

Remark. If A is a Dirichlet algebra, then the usage of (11) in the above proof can be avoided as follows. Pick $f \in A$ such that $\mathrm{Re} f > 1$ on X, $\mathrm{Re} f > 1/\varepsilon$ on K, and $\int f\,dm < (1 - 2^{-1}\varepsilon^2)^{-1}$. If $R > 0$ is large enough, then $f^{-1} = R^{-1}\{1 - (1 - R^{-1}f)\}^{-1}$ admits a uniformly convergent power series expansion in $1 - R^{-1}f$, so $g = f^{-1} \in A$. It is easy to check that g has the desired properties.

COROLLARY 4. *Let $\mu_s \in M(X)$ be singular with respect to m. Then there exists a sequence (h_n) in A_0 such that $\|h_n\|_X < 2$ for all n,*

$$\lim h_n = 1 \quad \mu_s\text{-a.e.} \quad \text{and} \quad \lim h_n = 0 \quad m\text{-a.e.}$$

Proof. Choose compact subsets $K_1 \subset K_2 \subset \ldots$ of X so that $m(K_n) = 0$ and $|\mu_s|(K_n) > \|\mu_s\| - 1/n$ for all n. By Theorem 3, there exist $g_n \in A$ such that $|g_n| < 1$ on X, $|g_n| < 1/n$ on K_n and $\int |g_n - 1|\,dm < 1/n^2$. It will suffice to set $h_n = g_n - \int g_n\,dm$ for each n.

COROLLARY 5 (cf. [6]). *Suppose $\mu \in A_0^{\perp}$, i.e., μ is a measure in $M(X)$ which annihilates A_0. Then the singular part of μ annihilates A.*

Proof. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to m. Choose $(h_n) \subset A_0$ as in Corollary 4. If $f \in A$, then $fh_n \in A_0$, so

$$\int fh_n\,d\mu_s = -\int fh_n\,d\mu_a \quad \forall n \in \mathbb{N} \tag{12}$$

by hypothesis. Apply the Lebesgue convergence theorem to obtain $\int f\,d\mu_s = 0$.

COROLLARY 6. *Suppose that $C(X)$ contains a uniformly dense subset D such that*

$$(*) \qquad \forall f \in D,\ \exists p \in \mathbb{N} \quad \text{such that} \quad fh^p \in A_0 \quad \forall h \in A_0.$$

Then $\mu \in A_0^{\perp}$ implies $\mu \ll m$.

Proof. Let $\mu \in A_0^{\perp}$ and $f \in D$ be given. Choose $p \in \mathbb{N}$ as in $(*)$. Then (12) holds with h_n replaced by h_n^p, so $\int f\,d\mu_s = 0$. Since D is uniformly dense in $C(X)$, we conclude that $\mu_s = 0$.

COROLLARY 7 (The F. and M. Riesz Theorem). *If $\mu \in M(\mathbb{T})$ and $\hat{\mu}(n) < 0$ for all negative integers n, then μ is absolutely continuous.*

Proof. In Corollary 6, choose $A = \{f \in C(\mathbb{T}) : \hat{f}(n) = 0 \text{ for all } n < 0\}$, $m = m_{\mathbb{T}}$, and $D =$ the set of all trigonometric polynomials.

REFERENCES

[1] R. Arens and I. M. Singer, *Function values as boundary integrals*, Proc. Amer. Math. Soc. 5 (1954), 735–745.

[2] E. Bishop, *A general Rudin–Carleson theorem*, ibid. 13 (1962), 140–143.

[3] R. Doss, *On the transform of a singular or absolutely continuous measure*, ibid. 19 (1968), 361–363.

[4] —, *Elementary proof of the Rudin–Carleson theorem and the F. and M. Riesz theorems*, ibid. 82 (1981), 599–602.

[5] T. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs 1969.

[6] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. 99 (1958), 165–202.

[7] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs 1962.
[8] —, *Analytic functions and logmodular Banach algebras*, Acta Math. 108 (1962), 271–317.
[9] W. Rudin, *Fourier Analysis on Groups*, Interscience, New York 1962.
[10] J. Werner, *Seminar über Funktionen-Algebren*, Lecture Notes in Math. 1, Springer, 1964.

DEPARTMENT OF MATHEMATICS
KANSAS STATE UNIVERSITY
MANHATTAN, KANSAS 66506, U.S.A.

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