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A REMARK ON ELEMENTARY PSEUDOGROUPS

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Dubikajtis [2] has called an *elementary pseudogroup* the system C defined by following axioms:

AXIOM 1. $\bigwedge_{x,y}\bigvee_{1z} x\cdot y = z$, where $\bigvee_{1z}$ means that there exists one and only one z.

AXIOM 2. $x\cdot(y\cdot z) = (x\cdot y)\cdot z$.

AXIOM 3. $x, y\in C_0 \Rightarrow x\cdot y\in C_0$, where C_0 is a fixed subset of C.

AXIOM 4. $x, y\in C_0 \Rightarrow x\cdot y = y\cdot x$.

AXIOM 5. $x\in C_0 \Rightarrow x\cdot x = x$.

DEFINITION 1. $y = ax \Leftrightarrow y\in C_0 \wedge x\cdot y = x \wedge \bigwedge_{z\in C_0}(x\cdot z = x \Rightarrow y\cdot z = y)$.

DEFINITION 2. $y = \beta x \Leftrightarrow y\in C_0 \wedge y\cdot x = x \wedge \bigwedge_{z\in C_0}(z\cdot x = x \Rightarrow z\cdot y = y)$.

AXIOM 6. $\bigwedge_x\bigvee_y y = ax \wedge \bigwedge_x\bigvee_y y = \beta x$.

AXIOM 7. $\bigwedge_x\bigvee_y (y\cdot x = ax \wedge x\cdot y = \beta x)$.

AXIOM 8. $x = y\cdot ax \Leftrightarrow x = \beta x\cdot y$.

Maniakowski [5] has proved that the system of above axioms of elementary pseudogroup is equivalent to the following one:

AXIOM 1′. $\bigwedge_{x,y}\bigvee_{1z} x\cdot y = z$.

AXIOM 2′. $x\cdot(y\cdot z) = (x\cdot y)\cdot z$.

AXIOM 3′. $x, y\in C_0 \Rightarrow x\cdot y = y\cdot x$.

AXIOM 4′. $x\in C_0 \Rightarrow x\cdot x = x$.

AXIOM 5′. $\bigwedge_x\bigvee_u [u\in C_0 \wedge x\cdot u = x \wedge \bigwedge_z (x\cdot z = x \Rightarrow u\cdot z = u)] \wedge$
$\wedge \bigwedge_x\bigvee_v [v\in C_0 \wedge v\cdot x = x \wedge \bigwedge_z (z\cdot x = x \Rightarrow z\cdot v = v)]$.

AXIOM 6′. $\bigwedge_x\bigvee_y x\cdot y\cdot z = x$.

In this note we prove the following

THEOREM. *Maniakowski's system, and so the notion of elementary pseudogroup, is equivalent to the notion of inverse semigroup with the set C_0 of idempotents.*

Proof. Let C be a Maniakowski's system. By Axioms 1′, 2′, and 6′,

C is a regular semigroup. In the proof of Lemma 1 in [5] it was shown that $x \cdot x = x$ implies $x \epsilon C_0$. Hence remove from Axiom 4′ we infer that C_0 is the set of idempotents of C. Therefore, by Axiom 3′, C is an inverse semigroup. (For the definition and properties of inverse semigroup, cf. [1] or [4].) Conversely, let C be an inverse semigroup with the set C_0 of idempotents. It is trivial that C satisfies all axioms of Maniakowski's system except, perhaps, Axiom 5′. As for Axiom 5′, it is easily seen that $\bar{x} \cdot x$ and $x \cdot \bar{x}$, where $\bar{x}$ is the inverse of x, satisfy requirements for u and v, respectively.

Appendix. I wish to express my hearty thanks to Prof. L. Dubikajtis for his various informations on the subject. Let me also remark that the same result has been obtained (but not published) by B. M. Szajn.

REFERENCES

[1] A. H. Clifford and G. B. Preston, *Algebraic theory of semigroups I*, American Mathematical Society Surveys 7 (1961).

[2] L. Dubikajtis, *Certaines extensions de la notion de groupoïde inductif et de celle de pseudogroupe*, Colloquium Mathematicum 12 (1964), p. 163-185.

[3] — and P. Jarek, *Pseudogroupe élémentaire commutatif et semi-groupe régulier commutatif*, ibidem 12 (1964), p. 187-193.

[4] Е. С. Ляпин, *Полугруппы*, Москва 1960.

[5] F. Maniakowski, *Sur les axiomes du pseudogroupe*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 12 (1964), p. 197-201.

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