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A MEASURE INDUCED METRIC TOPOLOGY FOR A BANACH ALGEBRA *

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Introduction. By means of a measure defined on the maximal ideal space of a Banach algebra X, the author is able to define a new metric on X. Denoting this metric by d, the resultant metric space X_d is investigated from two different points of view. In the first three sections the interplay between the algebraic structure of X and the metric topology is studied. It is shown, for example, that X_d is a bounded topological ring. In addition, the topological properties of certain algebraic subsets (e.g., the units, the idempotents, the maximal ideals) are contrasted in the Banach algebra topology and the metric topology. For instance, while the units are always open in the Banach algebra topology, they are open in X_d iff d is trivial. Furthermore, we present an example in which there are no maximal ideals which are closed in X_d.

Questions relating to completeness and convergence are dealt with in sections 4 and 5. We show that if the maximal ideal space is denumberable, X_d is incomplete. A strengthening of the concept of convergence in measure, called convergence strongly in measure, is defined on a general measure space. It is then shown that a sequence converges in X_d iff the corresponding sequence of Gelfand functions converges strongly in measure. In the sixth and final section it is shown that X_d is locally compact iff d is trivial.

1. Preliminairies. Let $X = (X, \| \ \|)$ be a complex commutative semi-simple Banach algebra with identity e such that $\|e\| = 1$. Furthermore, let $\mathscr{M}$ be the maximal ideal space of X with the Gelfand topology on it. Denote by m a probability measure on $\mathscr{M}$ which satisfies the additional condition of being positive on non-empty open sets.

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THEOREM 1.1. *For x, y in X, define $d(x, y) = m(\hat{x} \neq \hat{y})$, where $\hat{x}$ and $\hat{y}$ are the Gelfand functions associated with x and y, respectively, and $(\hat{x} \neq \hat{y})$ is the subset of $\mathscr{M}$, where $\hat{x}$ and $\hat{y}$ are unequal. Then d is a metric on X.*

Proof. Suppose x and y are such that $d(x, y) = m(\hat{x} \neq \hat{y}) = 0$. Since $\hat{x}$ and $\hat{y}$ are continuous, $(\hat{x} \neq \hat{y})$ is open in $\mathscr{M}$, and since m is positive on non-empty open sets, $m(\hat{x} \neq \hat{y}) = 0 \Rightarrow (\hat{x} \neq \hat{y}) = \emptyset \Rightarrow \hat{x} = \hat{y} \Rightarrow$ (since X is semi-simple) $x = y$.

The triangle inequality is a consequence of the following set relation which holds for any x, y, z in X:

$$(\hat{x} \neq \hat{y}) \subset (\hat{x} \neq \hat{z}) \cup (\hat{z} \neq \hat{y}).$$

THEOREM 1.2. *Letting $d(x, 0) = |x|$, we have*

$$\text{(a)} \qquad |x+y| \leqslant |x| + |y|,$$

$$\text{(b)} \qquad |xy| \leqslant \min\{|x|, |y|\}.$$

Proof. Letting $z = x+y$ and $w = xy$, these inequalities are immediate consequences of the set relations

$$(\hat{z} \neq 0) = (\hat{x}+\hat{y} \neq 0) \subset (\hat{x} \neq 0) \cup (\hat{y} \neq 0),$$

$$(\hat{w} \neq 0) = (\hat{x}\hat{y} \neq 0) = (\hat{x} \neq 0) \cap (\hat{y} \neq 0) \subset \begin{cases} (\hat{x} \neq 0), \\ (\hat{y} \neq 0). \end{cases}$$

THEOREM 1.3. *X_d is a topological ring.*

Proof. The continuity of vector subtraction is a consequence of part (a) of theorem 1.2. The continuity of vector multiplication at an arbitrary point $z_0 = (x_0, y_0)$ follows from the set relation

$$(\hat{w} \neq \hat{w}_0) = (\hat{x}\hat{y} \neq \hat{x}_0\hat{y}_0) \subset (\hat{x} \neq \hat{x}_0) \cup (\hat{y} \neq \hat{y}_0),$$

where $w = xy$ and $w_0 = x_0 y_0$.

Remark. While X is a topological algebra in its Banach algebra topology, this is not the case in the metric topology. For if a is any non--zero scalar, $|ax| = |x|$ for all x in X.

2. Structure of X_d. As one might expect, or at least hope for, we can completely characterize X_d when X is finite dimensional.

THEOREM 2.1. *If X is finite dimensional, d is trivial.*

Proof. Since X_d is a topological ring, it suffices to produce an $\varepsilon > 0$ such that $|x| \geqslant \varepsilon$ for all $x \neq 0$. In the finite dimensional case, $\mathscr{M}$ is a finite discrete topological space. Taking

$$\varepsilon = \min\{m(M) \mid M \in \mathscr{M}\}$$

will do the trick.

THEOREM 2.2. (a) *The set of idempotents of X is closed in X_d.*
(b) *If X_d is complete, then U, the units of X, is closed in X_d.*

Proof. Since both proofs are similar, only the proof of (b) is presented. Let $x \in \overline{U}$ ($\overline{U}$ is the closure of U in X_d). Then there exists a sequence $\{x_n\}$ in U such that $x_n \to x$. Since $|x_n^{-1}| = |x_n|$ for each n, the sequence $\{x_n^{-1}\}$ is Cauchy and so there exists a $y \in X_d$ such that $x_n^{-1} \to y$. Then we have, for any n,

$$|xy - e| \leqslant |x(y - x_n^{-1})| + |x_n^{-1}(x - x_n)| \leqslant |y - x_n^{-1}| + |x - x_n|.$$

Since the right-hand side of this inequality can be made arbitrarily small by taking n sufficiently large, we must have

$$|xy - e| = 0 \Rightarrow xy = e \quad \text{or} \quad y = x^{-1} \Rightarrow x \in U.$$

Hence U is closed in X_d.

Definition 2.1. A commutative topological ring R is said to be *bounded* iff given any neighborhood V of 0, there exists a neighborhood W of 0 such that $WR \subset V$.

THEOREM 2.3. *X_d is a bounded topological ring.*

Proof. Referring to definition 2.1, let V be any neighborhood of 0 such that there exists an $\varepsilon > 0$ such that $S(0, \varepsilon) \subset V$, where $S(0, \varepsilon)$ denotes the spherical ball centered at 0 of radius ε. Since $|xy| \leqslant \min\{|x|, |y|\}$, we have $S(0, \varepsilon)X_d \subset S(0, \varepsilon) \subset V$. Thus, by taking $W = S(0, \varepsilon)$, we may conclude that X_d is bounded.

Jacobson [2] defines the concept of semi-simplicity for arbitrary rings. This definition collapses to the usual definition of semi-simplicity (i.e., intersection of the maximal ideals is the 0 only) in commutative rings with identity. Since the ideal structure of the Banach algebra X and the topological ring X_d coincide, we have

THEOREM 2.4. *X_d is a bounded semi-simple topological ring.*

THEOREM 2.5. *The units of X are open in X_d iff d is trivial.*

Proof. A bounded semi-simple ring in which the units are open is necessarily discrete (cf. [3], p. 158).

3. Ideals in X_d. In this section we present results pertaining to the topological properties of ideals, in particular maximal ideals, in X_d.

THEOREM 3.1. *Let I be an ideal. Then $\bar{I}$ is also an ideal.*

Proof. The proof of this result is word for word the same as the proof of the corresponding theorem in Banach algebra theory and is therefore omitted (cf. [1], p. 327).

THEOREM 3.2. *X_d always contains proper closed ideals.*

Proof. If d is trivial, there is nothing to prove. So assume d is not trivial. Then there exists an $x_0 \in X$ (necessarily a non-unit) such that $x_0 \neq 0$ and $|x_0| < 1$. Let $\varepsilon = |x_0|$ and let $I = (x_0)$ be the ideal generated by x_0. If $z \in I$, then there exists a $y \in X$ such that $z = x_0 y$, whence $|z| = |x_0 y| \leqslant |x_0|$ and, consequently, $I \subset S[0, \varepsilon]$. Thus $x \in \bar{I}$ implies $|x| \leqslant \varepsilon < 1$. Hence $\bar{I}$ is proper.

While the maximal ideals are necessarily closed in the Banach algebra topology, this need not be the case in the metric topology. In fact, we have the following suprising result:

THEOREM 3.3. *If $X = C[0, 1]$ and m is Lebesgue measure, then there are no closed maximal ideals in the resultant metric topological ring X_d.*

Proof. To begin with, we make the topological identification on $\mathscr{M}$ with $[0, 1]$ and, for each $f \in C[0, 1]$, $\hat{f} = f$. Let M be a maximal ideal. Then there exists an $x_0 \in [0, 1]$ such that we have $M = \{f \in X / f(x_0) = 0\}$. Let $e \in C[0, 1]$ be the identity function and let $\varepsilon > 0$ be given. Clearly, there exists an f in $C[0, 1]$ such that $f(x_0) = 0$ and $d(f, e) \doteq m(f \neq e) < \varepsilon$. Thus $e \in \bar{M}$, but $e \notin M$. Thus M is not closed in X_d. (By a straightforward generalization of this argument, it becomes apparent that $\bar{M} = C[0, 1]$, i.e., the maximal ideals of $C[0, 1]$ are dense in X_d.)

Even though the above shows that a maximal ideal need not be closed in X_d, there is a very simple sufficient condition which will guarantee a maximal ideal to be closed.

THEOREM 3.4. *If M is an atom of m, then M is both open and closed in X_d.*

Proof. Recall that an atom is a point of positive measure.

(a) Let $x_0 \in M$ ($\Rightarrow \hat{x}_0(M) = 0$) and let $\varepsilon = m(\{M\})$. We show that $S(x_0, \varepsilon) \subset M$ which implies that M is open in X_d. Let $x \in S(x_0, \varepsilon)$. Then $d(x, x_0) < \varepsilon$, and so $m(\hat{x} \neq \hat{x}_0) < \varepsilon = m(\{M\})$. Now $x \notin M = \hat{x}(M) \neq 0$ $\Rightarrow M \in (\hat{x} \neq \hat{x}_0) \Rightarrow m(\hat{x} \neq \hat{x}_0) \geqslant m(\{M\})$ or $d(x, x_0) \geqslant \varepsilon$. Hence $x \in M$.

(b) While it can be shown directly, it follows from (a) and a general result from the theory of topological groups (open subgroups are necessarily closed) that M is closed in X_d.

COROLLARY 3.1. *If m has an atom, X_d is not connected.*

4. Convergence in X_d. We begin with a strengthening of the concept of convergence in measure.

Definition 4.1. Let $(\Omega, \mathscr{S}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of measurable functions and let f be measurable also. We say f_n converges to f *strongly in measure*, and write $f_n \to f$ (s.m.), iff for each $\varepsilon > 0$ there exists an integer N such that $\mu(|f_n - f| \geqslant \sigma) < \varepsilon$ for $n > N$ and all $\sigma > 0$.

The reader should have no trouble in constructing simple examples to show that this definition is not void. While convergence strongly in measure is obviously stronger than convergence in measure, it is still not strong enough to imply convergence almost everywhere. An example illustrating this may be found in [4], p. 96. In the same vein, convergence almost everywhere does *not* imply convergence strongly in measure. (Take $\Omega = [0, 1]$, μ to be Lebesgue measure, and $f_n(x) = x/n$.)

THEOREM 4.1. *$x_n \to x$ in X_d iff $\hat{x}_n \to \hat{x}$ (s. m.).*

Proof. (i) Let $\varepsilon > 0$ be given. Then $x_n \to x$ in X_d, and so there exists a positive integer N such that $n > N \Rightarrow m(\hat{x} \neq \hat{x}_n) < \varepsilon$.

But, for any $\sigma > 0$, $(|\hat{x}-\hat{x}_n| \geqslant \sigma) \subset (\hat{x} \neq \hat{x}_n)$, so for any $\sigma > 0$ we have $n > N \Rightarrow m(|\hat{x}-\hat{x}_n| \geqslant \sigma) < \varepsilon$.

Thus $\hat{x}_n \to \hat{x}$ (s.m.).

(ii) For each positive integer n we have

$$(\hat{x} \neq \hat{x}_n) = \bigcup_k (|\hat{x}-\hat{x}_n| \geqslant 1/k)$$

and the union is increasing. Let $\varepsilon > 0$ be given. Then $\hat{x}_n \to \hat{x}$ (s.m.), and so there exists a positive integer N such that

$$n > N \Rightarrow m(|\hat{x}-\hat{x}_n| \geqslant 1/k) < \varepsilon \quad \text{for all } k = 1, 2, \ldots$$

Hence

$$\begin{aligned} n > N \Rightarrow d(x, x_n) &= m(\hat{x} \neq \hat{x}_n) = m\big[\bigcup_k (|\hat{x}-\hat{x}_n| \geqslant 1/k)\big] \\ &= \lim_{k\to\infty} m(|\hat{x}-\hat{x}_n| \geqslant 1/k) \leqslant \varepsilon \Rightarrow x_n \to x \text{ in } X_d. \end{aligned}$$

COROLLARY 4.1. *$x_n \to x$ in X_d implies $\hat{x}_n \to \hat{x}$ (in measure) and there exists a subsequence $\{x_{n_i}\}$ such that $\hat{x}_{n_i} \to \hat{x}$ almost everywhere.*

It is now time to consider an example. With the aid of the previous theorem, we prove the following

CONTENTION. *Let $X = C[0, 1]$ and let m be Lebesgue measure. Then the resultant metric space X_d is incomplete.*

We prove this by actually producing a Cauchy sequence which does not converge. As usual, we identify $\mathscr{M}$ topologically with $[0, 1]$ and $\hat{f} = f$ for all f in $C[0, 1]$. Consider the sequence $\{f_n\}$ in $C[0, 1]$ defined as

$$f_n(x) = \begin{cases} 0 & \text{for } 0 \leqslant x \leqslant 1/2 - 1/(n+2), \\ n(x-1/2)+1 & \text{for } 1/2 - 1/(n+2) < x < 1/2, \\ 1 & \text{for } 1/2 \leqslant x \leqslant 1. \end{cases}$$

Let $n < m$. Then we have

$$(\hat{f}_n \neq \hat{f}_m) = (f_n \neq f_m) = \big(1/2 - 1/(n+2), 1/2\big) \Rightarrow d(f_n, f_m) = 1/(n+2).$$

Hence the sequence $\{f_n\}$ is Cauchy in the metric d. Now the sequence $\{f_n\}$ converges pointwise to the step function f, where

$$f(x) = \begin{cases} 0 & \text{for } 0 \leqslant x < 1/2, \\ 1 & \text{for } 1/2 \leqslant x \leqslant 1. \end{cases}$$

There cannot exist a g in $C[0, 1]$ such that $f_n \to g$ in X_d. For suppose there did. Then we would have $f_n \to g$ (in measure) from which it follows that we must have $g = f$ almost everywhere, which is impossible since g is continuous.

5. Completeness of X_d. A more general result with respect to the completeness (or, to be specific, lack of completeness) of X_d.

LEMMA 5.1. *Let Y be a Banach algebra with identity, and let $\{M_1, \ldots, M_n\}$ be a finite collection of distinct maximal ideals in Y. Then if $\{a_1, \ldots, a_n\}$ is any finite collection of complex numbers, there exists a $y \in Y$ such that $\hat{y}(M_i) = a_i$ for all $i = 1, \ldots, n$* (cf. [5], p. 38).

LEMMA 5.2. *m has at most countably many atoms.*

Proof. Since m is finite, there are at most finitely many atoms with measure greater than $1/n$ for each positive integer n.

THEOREM 5.1. *Let $\mathscr{M}$ be infinite. If $\mathscr{A} \subset \mathscr{M}$ is the set of atoms of m and $m(\mathscr{A}) = 1$, then X_d is incomplete.*

Proof. We show first that $\mathscr{A}$ is infinite. Suppose it is not. Then let $\mathscr{A} = \{A_1, \ldots, A_n\}$ and $M \notin \mathscr{A}$. Since $\mathscr{M}$ is Hausdorff, for each i, $1 \leqslant i \leqslant n$, there exist disjoint open sets $\mathscr{U}_i$, $\mathscr{V}_i$ in $\mathscr{M}$ such that $M \in \mathscr{U}_i$ and $A_i \in \mathscr{V}_i$. Let $\mathscr{U} = \bigcap \mathscr{V}_i$. Then $\mathscr{U}$ is a non-empty open set such that $m(\mathscr{U}) = 0$ which is impossible.

Thus let $\mathscr{A} = \{A_1, \ldots, A_n, \ldots\}$. We prove that X_d is incomplete by producing a Cauchy sequence which does not converge. Let $\varepsilon > 0$ be given and let $\mathscr{A}_n = \{A_1, \ldots, A_n\}$. Then, $1 = m(\mathscr{A}) = \lim m(\mathscr{A}_n)$, and so there exists an $N > 0$ such that $n \geqslant N \Rightarrow m(\mathscr{A}_n) > 1 - \varepsilon$ or, equivalently, $m(\mathscr{A}_n^c) < \varepsilon$, where $\mathscr{A}_n^c$ is the complement of $\mathscr{A}_n$.

Now, by virtue of lemma 5.1, there exists a sequence $\{x_n\}$ such that $\hat{x}_n(A_i) = i$ for $1 \leqslant i \leqslant n$.

Take $n, m > N$. Then we have

$$(\hat{x}_n \neq \hat{x}_m) \subset \mathscr{A}_N^c \Rightarrow d(x_n, x_m) = m(\hat{x}_n \neq \hat{x}_m) \leqslant m(\mathscr{A}_N^c) < \varepsilon.$$

Thus $\{x_n\}$ is Cauchy in X_d.

Suppose there exists an x in X_d such that $x_n \to x$ in X_d. Then, by corollary 4.1, there exists a subsequence $\{x_{n_i}\}$ such that $\hat{x}_{n_i} \to \hat{x}$ almost everywhere. Since $m(A_k) > 0$ for all A_k, and $\hat{x}_{n_i}(A_k) \to \hat{x}(A_k)$ for all $k = 1, 2, \ldots$, $\hat{x}_{n_i}(A_k) \to k$, so $\hat{x}(A_k) = k$ for all $k = 1, 2, \ldots$ We have now arrived at a contradiction, since we must have $|x(M)| \leqslant \|x\|$ for all $M \in \mathscr{V}_i$. Thus X_d is incomplete.

COROLLARY 5.1. *If $\mathscr{M}$ is denumerable (countable, but not finite), then X_d is incomplete.*

Proof. If $\mathscr{M}$ is denumerable, the hypothese of the theorem is clearly satisfied.

6. Local compactness. First, a few preliminary results.

THEOREM 6.1. (a) *A bounded locally compact ring has a system of ideal neighborhoods of* 0 (cf. [3], p. 160).

(b) *For any neighborhood U of* 0 *in a compact ring R, there exists a positive integer N such that $R^n \subset U$ for $n > N$* (cf. [3], p. 163).

(c) *A compact semi-simple ring must have an identity* (cf. [3], p. 163).

(d) *An ideal in a semi-simple commutative ring is also semi-simple* (cf. [2], p. 314).

THEOREM 6.2. *X_d is locally compact iff d is trivial.*

Proof. Suppose X_d is locally compact. Then there exists a neighborhood O of 0 such that $\bar{O}$ is compact. By theorem 6.1 (a) there exists an ideal $I \subset O$ such that, by theorem 3.1, $\bar{I}$ is a compact ideal. From theorem 6.1 (d) we conclude that $\bar{I}$ is semi-simple and hence, by theorem 6.1 (c), must have an identity (not necessarily the identity e of X_d). Since $\bar{I}$ contains an identity, $(\bar{I})^n = \bar{I}$ for all positive integers n. From theorem 6.1 (b) we get $\bar{I} \subset S(0, \varepsilon)$ for all $\varepsilon > 0$, whence $\bar{I} = \{0\}$. Therefore $I = \{0\}$, and so d is trivial, because I is a neighborhood of 0.

COROLLARY 6.1. *X_d is never compact.*

REFERENCES

[1] G. Bachman and L. Narici, *Functional analysis*, New York and London 1966.

[2] N. Jacobson, *The radical and semi-simplicity for arbitrary rings*, American Journal of Mathematics 67 (1945), p. 300-320.

[3] I. Kaplansky, *Topological rings*, ibidem 69 (1947), p. 153-183.

[4] I. P. Natanson, *Theory of functions of a real variable*, Vol. 1, New York 1961.

[5] C. Suffel, Ph. D. dissertation, Polytechnic Institute of Brooklyn, Brooklyn, New York 1969.

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