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STABLE DERIVED FUNCTORS OF THE SECOND SYMMETRIC POWER FUNCTOR, SECOND EXTERIOR POWER FUNCTOR AND WHITEHEAD GAMMA FUNCTOR

BY

DANIEL SIMSON (TORUŃ)

Let $\mathscr{A}$ and $\mathscr{A}'$ be Abelian categories, let $\mathscr{A}$ have enough projectives, and let $T\colon \mathscr{A} \to \mathscr{A}'$ be a covariant functor such that $T(0) = 0$. In [2], Dold and Puppe defined *left derived functors*

$$L_q T(\cdot, n)\colon \mathscr{A} \to \mathscr{A}', \quad n, q \in Z, n \geqslant 0,$$

and *natural transformations of functors*

$$\sigma_q^n\colon L_q T(\cdot, n) \to L_{q+1} T(\cdot, n+1).$$

A *q-th left stable derived functor* of the functor T is the functor

$$L_q^s T = \varinjlim_n \{L_{q+n} T(\cdot, n), \sigma_{q+n}^n\}.$$

The functors $L_q^s T$ are additive and form a connected and exact sequence of covariant functors provided T is either a direct sum or a direct product of functors of finite degree (see [5] and [6]).

The main purpose of this paper is a computation of the left stable derived functors of the second symmetric power functor SP^2, the second exterior power functor Λ^2 and the J. H. C. Whitehead functor Γ.

It is proved that $L_q^s SP^2 \cong L_{q-1}^s \Lambda^2 \cong L_{q-2}^s \Gamma$ for any integer q and the values of $L_q^s SP^2$ on the category of all flat modules are computed. Moreover, it is shown that when 2 is a non-zero divisor in the ring R and either (i) w.gl.dim $R \leqslant 1$ or (ii) $r^2 - r \in 2R$ for any $r \in R$, then $L_q^s SP^2 \cong L_{q+2}^s SP^2$ for $q \geqslant 2$, $L_2^s SP^2 \cong U$ and $L_3^s SP^2 \cong L_1 U$, where U is a right exact functor defined in [6], Section 8 (see also Section 1 in this paper), and $L_1 U$ is its first left derived functor in the sense of [1]. As a corollary one obtains Theorem 3 in [5].

The results have been obtained in [6] in the case where the ring R satisfies condition (ii) above.

Throughout this paper R is a commutative ring with the identity element, *Mod-R* denotes the category of all unitary R-modules and *Fl-R* is the category of all flat R-modules.

1. The functors SP^2, Λ^2 and Γ. Recall that the *second symmetric power functor* SP^2: *Mod-R* $\to$ *Mod-R* and the *second exterior power functor* Λ^2: *Mod-R* $\to$ *Mod-R* are defined by

$$SP^2(M) = \otimes^2(M)/E(M), \quad \Lambda^2(M) = \otimes^2(M)/V(M),$$

where $E(M)$ is a submodule of $\otimes^2(M) = M \otimes_R M$ generated by all elements $m\otimes n - n\otimes m$, $m, n \in M$, and $V(M)$ is a submodule of $\otimes^2(M)$ generated by elements $m\otimes m$, $m \in M$. If f: $M \to N$ is a homomorphism of R-modules, then

$$SP^2(f)(m\vee m') = f(m)\vee f(m'), \qquad \Lambda^2(f)(m\wedge m') = f(m)\wedge f(m'),$$

where $m\vee m'$ and $m\wedge m'$ are images of $m\otimes m'$ by the natural epimorphisms $\otimes^2(M)\to SP^2(M)$ and $\otimes^2(M)\to \Lambda^2(M)$, respectively. Clearly, E and V are covariant functors (subfunctors of the second tensor power functor $\otimes^2$) and the homomorphisms

$$\Phi(M)\colon \Lambda^2(M) \to E(M)$$

given by $m\wedge m' \mapsto m\otimes m' - m'\otimes m$ define a natural transformation of functors Φ.

LEMMA 1.1. *Φ is a natural equivalence of functors on the category Fl-R.*

Proof. Since Λ^2 and E commute with direct limits and, by [4], any flat R-module is a direct limit of finitely generated free modules, it is sufficient to show that $\Phi(M)$ is an isomorphism whenever M is free and finitely generated. If $e_1, \ldots, e_n$ is a basis of M, then it is easy to check that $e_i\otimes e_j - e_j\otimes e_i$, $i<j$, is a basis of $E(M)$ and, therefore, $\Phi(M)$ is an isomorphism because the elements $e_i\wedge e_j$, $i<j$, form a basis of $\Lambda^2(M)$.

A function φ: $M\to N$ between R-modules M and N is said to be *quadratic* if $\varphi(rm) = r^2\varphi(m)$ for all $r\in R$ and $m \in M$ and if the associated symmetric function $\Delta\varphi$: $M\times M\to N$ defined by

$$\Delta\varphi(m, m') = \varphi(m+m') - \varphi(m) - \varphi(m'), \qquad m, m' \in M,$$

is R-bilinear. Let $\mathrm{Quad}_R(M, N)$ denote the R-module of all quadratic functions from M to N.

Following Whitehead [7], we define a *covariant functor*

$$\Gamma\colon \text{Mod-}R \to \text{Mod-}R$$

setting $\Gamma(M) = F(M)/F_0(M)$, where $F(M)$ is a free R-module freely generated by symbols $\omega(m)$, $m \in M$, and $F_0(M)$ is a submodule of $F(M)$ generated by elements of the form

$$\omega(rm) - r^2\omega(m),$$

$$\omega(m_1+m_2+m_3) - \omega(m_1+m_2) - \omega(m_1+m_3) - \omega(m_2+m_3) + \\ + \omega(m_1) + \omega(m_2) + \omega(m_3),$$

$$\omega(rm+m_1) - r\omega(m+m_1) - \omega(rm) + r\omega(m) - \omega(m_1) + r\omega(m_1),$$

$m, m_1, m_2, m_3 \in M$, $r \in R$ (see also [3] and [6], Section 8).

It is easy to check that the quadratic function $\gamma\colon M \to \Gamma(M)$ given by $\gamma(m) = \omega(m) + F_0(M)$ induces a natural isomorphism

$$\mathrm{Quad}_R(M, N) = \mathrm{Hom}_R(\Gamma(M), N).$$

It then follows that Γ commutes with direct limits (see Simson and Tyc [6], Corollary 8.3).

Consider a natural transformation of functors $\Psi\colon \Gamma \to V$ defined by $\gamma(m) \mapsto m \otimes m$.

LEMMA 1.2. *Ψ is a natural equivalence of functors on the category Fl-R.*

Proof. By arguments from the proof of Lemma 1.1, it is sufficient to show that $\Psi(M)$ is an isomorphism whenever M is free and finitely generated. Let $e_1, \ldots, e_n$ be a basis of M. It is easy to check that the elements $e_1 \otimes e_1, \ldots, e_n \otimes e_n$ and $e_i \otimes e_j + e_j \otimes e_i$, $i < j$, form a basis of $V(M)$. On the other hand, using the same type of arguments as in [7], p. 62, one shows that the elements $\gamma(e_1), \ldots, \gamma(e_n)$ and $\Delta\gamma(e_i, e_j)$, $i < j$, are free generators of $\Gamma(M)$. Hence $\Psi(M)$ is an isomorphism, and the proof of the lemma is complete.

Next we consider a natural transformation of functors $t\colon SP^2 \to \Gamma$ defined by $m \vee m' \mapsto \Delta\gamma(m, m')$ (see [3]) and define covariant functors U and W by $U = \mathrm{Coker}\, t$ and $W = \mathrm{Ker}\, t$, respectively. It is clear that U and W commute with direct limits.

A consequence of Lemmas 1.1 and 1.2 is the following

COROLLARY 1.3. *There is a diagram*

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & 0 & \to & \Lambda^2 \to \otimes^2 \to SP^2 \to 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \otimes^2 & & & & \\
 & & & & \uparrow & & & & \\
0 \to W \to SP^2 & \to & \Gamma & \to & U \to 0 & & & & \\
 & & \uparrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

with rows and column exact on the category Fl-R.

The following properties of the functors U and W are proved in [6], Sections 8 and 10:

(i) *U and W are additive functors.*

(ii) *U is right exact.*

(iii) *If M is a free module, then*

$$U(M) = \operatorname{Coker}(M \xrightarrow{2} M) \quad \text{and} \quad W(M) = \operatorname{Ker}(M \xrightarrow{2} M).$$

(iv) *If $r^2 - r \in 2R$ for all $r \in R$, then $U = (\cdot) \otimes_R R/2R$.*

2. Computations of stable derived functors of SP^2, Λ^2 and Γ. Let $\mathscr{A}$ and $\mathscr{A}'$ be Abelian categories, let $\mathscr{A}$ have enough projectives, and let $T\colon \mathscr{A} \to \mathscr{A}'$ be a covariant functor such that $T(0) = 0$. We shall need the following properties of the left stable derived functors $L_q^s T$ of the functor T:

(1) *$L_q^s T = 0$ for $q < 0$.*

(2) *$L_q^s T = L_q T$ if T is additive.*

(3) *$L_q^s T = 0$ whenever T is a composition*

$$\mathscr{A} \xrightarrow{\Delta} \mathscr{A} \times \mathscr{A} \xrightarrow{T'} \mathscr{A}',$$

where Δ is the diagonal functor and $T'(0, \cdot) = T'(\cdot, 0) = 0$.

(4) *Every p-exact sequence of functors*

$$0 \to T' \to T \to T'' \to 0$$

(i.e. exact on projective objects) induces a long exact sequence

$$\ldots \to L_q^s T' \to L_q^s T \to L_q^s T'' \to L_{q-1}^s T' \to \ldots$$

For details see [2] and [6].

Corollary 2.1. (a) *$L_q^s SP^2 \cong L_{q-1}^s \Lambda^2 \cong L_{q-2}^s \Gamma$ for any integer q.*

(b) $L_1^s SP^2 = L_0^s SP^2 = L_0^s \Lambda^2 = 0$.

Proof. Since $L_q^s \otimes^2 = 0$ by (3), then, in virtue of Corollary 1.3, (a) is an immediate consequence of (4). Assertion (b) follows from (a) and (1).

Proposition 2.2. (a) *$L_0^s \Gamma \cong U$.*

(b) *If 2 is a non-zero divisor in R, then $L_1^s \Gamma \cong L_1 U$.*

(c) *If $L_i W = L_{i+1} U = 0$ for $i \geqslant 1$ on a subcategory A of Mod-R, then there exist natural equivalences of functors $L_q^s SP^2 \cong L_{q+2}^s SP^2$ on A for $q \geqslant 2$.*

Proof. By (4), from the exact sequences

$$0 \to W \to SP^2 \to K \to 0 \quad \text{and} \quad 0 \to K \to \Gamma \to U \to 0$$

with $K = \operatorname{Im} t$, we derive the long exact sequences

$$\ldots \to L_q W \to L_q^s SP^2 \to L_q^s K \to L_{q-1} W \to \ldots$$

and

$$\ldots \to L_q^s K \to L_q^s \Gamma \to L_q U \to L_{q-1}^s K \to \ldots,$$

where $L_q^s(\cdot) = 0$ for $q < 0$ by (1), and $L_0^s SP^2 = L_1^s SP^2 = 0$ by Corollary 2.1. Hence $L_0^s K = 0$ and $L_1^s K \cong L_0 W$. Therefore, (a) and (b) follow since, by (iii), $L_0 W = 0$ provided 2 is a non-zero divisor in R.

To prove (c) observe that, in virtue of Corollary 2.1 and our assumption, the above long exact sequences yield natural equivalences of functors

$$L_q^s SP^2 \cong L_q^s K \cong L_q^s \Gamma \cong L_{q+2}^s SP^2, \qquad q \geqslant 2,$$

on the category A. This completes the proof of the proposition.

We are now able to prove the main results:

THEOREM 2.3. *There are natural equivalences of functors*

$$L_q^s SP^2 = \begin{cases} 0 & \text{for } q \leqslant 1, \\ U & \text{for } q = 2l,\ l \geqslant 1, \\ W & \text{for } q = 2l+1,\ l \geqslant 1, \end{cases}$$

regarded as functors from Fl-R to Mod-R.

Proof. Since, by [4], any flat module is a direct limit of free modules, the functors $L_i W$ and $L_i U$, $i \geqslant 1$, vanish on flat modules because they commute with direct limits. Thus, in virtue of Proposition 2.2, it is sufficient to compute the functor $L_3^s SP^2$, because $L_2^s SP^2 \cong L_0^s \Gamma \cong U$ by Corollary 2.1. For this purpose consider the long exact sequences from the proof of Proposition 2.2. Then, using Corollary 2.1 and preceding remarks, we derive the following natural equivalences of functors regarded as functors on Fl-R:

$$L_3^s SP^2 \cong L_1^s \Gamma \cong L_1^s K \cong L_0 W \cong W.$$

The proof is completed.

A ring R is said to be 2-*Boolean* if $R/2R$ is Boolean, i.e., $r^2 - r \in 2R$ for any $r \in R$.

THEOREM 2.4. *Suppose that* 2 *is a non-zero divisor in the ring* R. *If either* $\text{w.gl.dim} R \leqslant 1$ *or* R *is a* 2-*Boolean ring, then there are the following natural equivalences of functors*:

$$L_q^s SP^2 = L_{q-1}^s \Lambda^2 = L_{q-2}^s \Gamma = \begin{cases} 0 & \text{for } q \leqslant 1, \\ U & \text{for } q = 2l,\ l \geqslant 1, \\ L_1 U & \text{for } q = 2l+1,\ l \geqslant 1. \end{cases}$$

Proof. Since 2 is a non-zero divisor in R, we infer, by (iii) and (iv) in Section 1, that $L_i W = 0$ for all $i \geqslant 0$ and $L_2 U = \operatorname{Tor}_2^R(\cdot, R/2R) = 0$

whenever R is 2-Boolean. Clearly, if w.gl.dim$R \leqslant 1$, then $L_2 U$ also vanishes. Consequently, the theorem follows from Proposition 2.2 and Corollary 2.1.

3. Homology of SP^2, Λ^2 and Γ. A simplicial complex X is said to be *n-trivial* if it is homotopy equivalent to a simplicial complex Y with $Y_i = 0$ for $i < n$.

The following theorem is a generalization of Satz 12.1, in [2], for $n = 2$:

THEOREM 3.1. *Let X be a k-trivial simplicial complex consisting of flat R-modules. Then*

(a) $H_q SP^2(X) = H_{q-1}\Lambda^2(X)$ *and* $H_q\Lambda^2(X) = H_{q-1}\Gamma(X)$ *for* $q < 2k$;

(b) $H_q SP^2(X) = 0$ *for* $q < \min(2k, k+2)$;

(c) $H_q \Lambda^2(X) = 0$ *for* $q < \min(2k, k+1)$;

(d) $H_{k+2} SP^2(X) = U(H_k X)$ *whenever* $k > 2$;

(e) $H_{k+3} SP^2(X) = H_{k+1} U(X)$ *whenever* $k > 3$ *and* 2 *is a non-zero divisor in* R.

Proof. By Corollary 1.3, we have the following exact sequences:

$$0 \to \Lambda^2(X) \to \otimes^2(X) \to SP^2(X) \to 0,$$
$$0 \to \Gamma(X) \to \otimes^2(X) \to \Lambda^2(X) \to 0.$$

Since $H_q \otimes^2(X) = 0$ for $q < 2k$ by Hilfssatz 6.10 in [2], statement (a) follows from the long homology exact sequences induced by the above short sequences. Statements (b) and (c) follow from (a) and the obvious equality $H_q\Gamma(X) = 0$ for $q < k$. To prove (d) and (e) consider the exact homology sequences induced (see Corollary 1.3) by the exact sequences

$$0 \to W(X) \to SP^2(X) \to K(X) \to 0,$$
$$0 \to K(X) \to \Gamma(X) \to U(X) \to 0,$$

where $K(X) = \operatorname{Im} t(X)$. Since $H_{k-1}W(X) = 0$ and, by (b), $H_k SP^2(X) = 0$, we infer that $H_k K(X) = 0$. Hence

$$H_k \Gamma(X) = H_k U(X) = U(H_k X)$$

because U is right exact. Consequently, (d) follows from (a). Finally, suppose that $k > 3$ and 2 is a non-zero divisor in R. Then $W = 0$ by (iii) in Section 1 because W commutes with direct limits. Since $H_{k+1} SP^2(X) = 0$ by (c), statement (e) follows as (d) above and the proof is complete.

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INSTITUTE OF MATHEMATICS
NICHOLAS COPERNICUS UNIVERSITY, TORUŃ

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