COLLOQUIUM MATHEMATICUM

VOL. XXXV 1976 FASC. 1

ON SOME SUBSPACES OF THE HELLY SPACE

BY

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By virtue of being a simply defined function space, the Helly space seems to be the most natural example of a non-metrizable first countable compact Hausdorff space. The purpose of this paper is to indicate the considerable complexity of its topological structure. We shall show that all separable metrizable spaces are imbeddable into the Helly space and that they imbed as G_δ-subsets whenever they are compact.

Let H be the set of all non-decreasing functions from the closed unit interval $I = [0, 1]$ into itself. The set H endowed with the relative product topology is called the *Helly space*. It follows immediately that H is a compact Hausdorff space. For other elementary properties of H we refer the reader to [3], Chapter 5, Problem M, p. 164.

PROPOSITION 1. *A product of countably many copies of H is imbeddable into H.*

Proof. Let H_1 be the set of those $x \in H$ for which $x(t) = 1$ for all $t \in (1/2, 1]$. Let $t_n = 1-2^{-n}$ and let $\varphi_n(t) = t_{n-1}+(t_n-t_{n-1})t$, $t \in I$, $n = 1, 2, \ldots$ If

$$\xi = \{x_n\}_{n=1}^{\infty} \in \prod_{n=1}^{\infty} H_1,$$

set

$$\Phi(\xi)(t) = \varphi_n \circ x_n \circ \varphi_n^{-1}(t) \text{ for } t \in [t_{n-1}, t_n),\ n = 1, 2, \ldots, \quad \text{and} \quad \Phi(\xi)(1) = 1.$$

Then

$$\Phi\colon \prod_{n=1}^{\infty} H_1 \to H$$

is a continuous injection, and so it suffices to show that H_1 is homeomorphic to H. This homeomorphism is realized by a continuous bijection $\Psi\colon H \to H_1$ defined, if $x \in H$, by $\Psi(x)(t) = x(2t)$ for $t \in [0, 1/2]$ and by $\Psi(x)(t) = 1$ for $t \in (1/2, 1]$.

COROLLARY. *Every separable metrizable space is imbeddable into H.*

Indeed, by Proposition 1, and [2], Chapter 9, 8.4, p. 193, the Hilbert cube can be imbedded into H.

For $A \subset H$ we denote by D_A the set of all $t \in I$ at which some $x \in A$ is discontinuous.

PROPOSITION 2. *A set $A \subset H$ is separable and metrizable if and only if D_A is countable.*

Proof. (a) Suppose that D_A is countable. Let Q be the set of all rational numbers from I and let $R = D_A \cup Q = \{t_n\}_{n=1}^{\infty}$. For $x, y \in H$ set

$$\varrho(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x(t_n) - y(t_n)|.$$

Clearly, ϱ is a continuous pseudometric on H which is a metric on A. Thus to prove the metrizability of A we only need to show that the ϱ-topology on A is finer than the relative product topology. Choose $x \in A$, $t \in I$, $\varepsilon > 0$, and let

$$U = \{y \in A: |x(t) - y(t)| < \varepsilon\}.$$

If $t \in D_A$, then $t = t_k$ for some integer $k \geqslant 1$. Letting

$$V = \{y \in A: \varrho(x, y) < \varepsilon \cdot 2^{-k}\},$$

we have $V \subset U$. If $t \notin D_A$, then x is continuous at t and we can find $t_m, t_n \in Q$ such that $t_m < t < t_n$ and

$$x(t) - \varepsilon/2 < x(t_m) \leqslant x(t_n) < x(t) + \varepsilon/2.$$

Let $s = 1 + \max(m, n)$ and $\varrho(x, y) < \varepsilon \cdot 2^{-s}$. Then

$$x(t_m) - \varepsilon/2 < y(t_m) \leqslant y(t) \leqslant y(t_n) < x(t_n) + \varepsilon/2,$$

and hence $x(t) - \varepsilon < y(t) < x(t) + \varepsilon$. Thus, letting

$$V = \{y \in A: \varrho(x, y) < \varepsilon \cdot 2^{-s}\},$$

we have again $V \subset U$. Let A_1 be the set of all $x \in H$ for which $D_{\{x\}} \subset D_A$. Since $D_A = D_{A_1}$, A_1 is also metrizable by ϱ. Let B be the set of all continuous piecewise linear functions $x \in H$ such that $[t, x(t)] \in Q \times Q$ whenever x has a corner at t. Then $B \subset A_1$ and $B^- = H$. Thus A_1 is separable and so is A; for A_1 is metrizable.

(b) Suppose that D_A is uncountable. Since each $x \in H$ has only countably many discontinuities, there are an uncountable subset $D_1 \subset D_A$ and an injective map $t \mapsto x_t$ from D_1 into A such that x_t has a discontinuity at t. Then we can find an uncountable set $D_2 \subset D_1$ and $\varepsilon > 0$ such that $x_t(t+) - x_t(t-) > \varepsilon$ for all $t \in D_2$. Finally, there are an uncountable set $D_3 \subset D_2$ and $a, b \in (0, 1)$ such that

$$x_t(t-) < a < b < x_t(t+) \quad \text{for all } t \in D_3.$$

With no loss of generality we may also assume that, e.g., $x_t(t) < b$ for all $t \in D_3$. Let $A_0 = \{x_t\colon t \in D_3\}$. Considering the family of non-empty open sets

$$U_t = \{x \in A_0\colon 0 \leqslant x(t) < b\}, \qquad t \in D_3,$$

it is clear that A_0 does not have a countable base.

For $x, y \in H$ write $x \sim y$ whenever $x(t-) = y(t-)$ and $x(t+) = y(t+)$ for all $t \in I$. Clearly, $x \sim y$ if and only if x and y agree at all points $t \in I$ at which they are both continuous. If $T \subset I$, write $x \sim_T y$ whenever $x \sim y$ and $x(t) = y(t)$ for all $t \in T$. Obviously, $\sim$ and $\sim_T$ are equivalence relations on H, and we denote by $[x]$ and $[x]_T$ the corresponding equivalence classes of $x \in H$. For $A \subset H$, let

$$[A] = \bigcup_{x \in A} [x] \quad \text{and} \quad [A]_T = \bigcup_{x \in A} [x]_T.$$

We have $[A]_\emptyset = [A]$, $[A]_{D_A} = A$, and $[A]_S \subset [A]_T$ whenever $T \subset S$.

LEMMA 1. *If $A \subset H$ and $S, T \subset I$, then $\big[[A]_S\big]_T = [A]_{S \cap T}$.*

Proof. (a) If $x \in \big[[A]_S\big]_T$, then there are $y \in [A]_S$ and $z \in A$ such that

$$x \sim_T y \quad \text{and} \quad y \sim_S z.$$

Hence $x \sim_{S \cap T} z$, and so $x \in [A]_{S \cap T}$.

(b) If $x \in [A]_{S \cap T}$, then there is a $z \in A$ such that $x \sim_{S \cap T} z$. For $t \in I$, set $y(t) = x(t)$ whenever x is continuous at t or $t \in T$, and $y(t) = z(t)$ otherwise. Let $t_1, t_2 \in I$, $t_1 < t_2$, and choose $t_0 \in (t_1, t_2)$ such that x is continuous at t_0. Then $x(t_0) = z(t_0) = y(t_0)$, and hence $y(t_1) \leqslant y(t_0) \leqslant y(t_2)$. Thus $y \in H$ and, clearly, $y \sim_T x$ and $y \sim z$. Choose $t \in S$. If x is continuous at t or $t \in T$, then $y(t) = x(t) = z(t)$; otherwise, $y(t) = z(t)$. Hence $y \sim_S z$ and $x \in \big[[A]_S\big]_T$.

Choose $T \subset I$. Since the functions $x \mapsto x(t-)$ and $x \mapsto x(t+)$, $t \in I$, are lower and upper semicontinuous, respectively, on H, it easily follows that the quotient space $H/\sim_T$ is Hausdorff. Thus, by [2], Chapter 6, 4.2 (1), p. 125, if $A \subset H$ is compact, so is $[A]_T$.

LEMMA 2. *Let $A, G \subset H$, A compact and G open. If $A \subset G$, then also $[A]_T \subset G$ for some finite set $T \subset I$.*

Proof. The set G is a union of open rectangles

$$G(t_1, \ldots, t_n;\ U_1, \ldots, U_n) = \{x \in H\colon x(t_i) \in U_i,\ i = 1, \ldots, n\},$$

where $t_i \in I$ and U_i, $i = 1, \ldots, n$, are open subsets of I. Since A is compact, it is covered by finitely many of these open rectangles, say

$$G(t_1^j, \ldots, t_{n_j}^j;\ U_1^j, \ldots, U_{n_j}^j), \qquad j = 1, \ldots, k.$$

Hence, if

$$T = \{t_i^j\colon i = 1, \ldots, n_j;\ j = 1, \ldots, k\},$$

then $[A]_T \subset G$.

PROPOSITION 3. *A compact set $A \subset H$ is G_δ if and only if $A = [A]_T$ for some countable set $T \subset I$.*

Proof. (a) Let

$$A = \bigcap_{n=1}^{\infty} G_n,$$

where $G_n \subset H$ are open sets. By Lemma 2, there are finite sets $T_n \subset I$ such that $[A]_{T_n} \subset G_n$. Letting

$$T = \bigcup_{n=1}^{\infty} T_n,$$

we have $A \subset [A]_T \subset [A]_{T_n} \subset G_n$, $n = 1, 2, \ldots$, and so $A = [A]_T$.

(b) Let $A = [A]_T$ for a countable set $T \subset I$ and let $S = \{t_n\}_{n=1}^{\infty}$ be a countable dense subset of I which contains T. For $x, y \in H$, set

$$\sigma(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x(t_n) - y(t_n)|.$$

It is easy to see that σ is a continuous pseudometric on H such that $\sigma(x, y) = 0$ if and only if $x \sim_S y$. If $\sigma(x, A) = \inf\{\sigma(x, y)\colon y \in A\}$, then $x \mapsto \sigma(x, A)$ is a continuous function on H. Hence $A_\sigma = \{x \in H\colon \sigma(x, A) = 0\}$ is a closed G_δ-subset of H and it follows easily from the compactness of A that $A_\sigma = [A]_S$. By Lemma 1,

$$[A]_S = \big[[A]_T\big]_S = [A]_{T \cap S} = [A]_T = A.$$

COROLLARY. *Compact metrizable subspaces of H are G_δ.*

Example 1. Let $A \subset H$ consist of all $x \in H$ which take values in $\{0, 1\}$. Then A is closed and it is easy to see that A is also perfectly normal. In fact, A is homeomorphic to the space A_7 from [1], Chapter 5, Section 1 (the latter space is sometimes quoted under the name "two arrows"). According to Proposition 3, A is not G_δ. With no difficulty one can also observe that $[A]$, which is G_δ, is homeomorphic to the lexicographically ordered square (see [3], Chapter 5, Problem J).

Example 2 (1). Let A be as in Example 1 and let $X = [A] \times [A] - - A \times A$. Then X is a locally compact Hausdorff space. We show that X is metacompact but not paracompact.

(a) Since

$$X = \{([A] - A) \times [A]\} \cup \{[A] \times ([A] - A)\},$$

where both summands are paracompact and open in X, it follows that X is metacompact.

(1) With respect to this example the author is obliged to G. Gruenhage for some stimulating discussions.

(b) Since $[A]$ is homeomorphic to the lexicographically ordered square, $[A]$ is connected. Choose $x_0 \in [A]-A$. If $(x, y) \in X$, then, e.g., $x \notin A$. The set $(\{x\} \times [A]) \cup ([A] \times \{x_0\})$ is a connected subset of X containing both (x_0, x_0) and (x, y). Therefore, X is connected and so, according to [2], Chapter 11, 7.3, p. 241, it suffices to show that X is not σ-compact. Let $B \subset H$ consist of all $x \in H$ for which $x(t) \in \{0, 1/2\}$ if $t \in [0, 1/3]$, $x(t) = 1/2$ if $t \in (1/3, 2/3)$, and $x(t) \in \{1/2, 1\}$ if $t \in [2/3, 1]$. It is easy to see that $A \times A$ and X are homeomorphic to B and $[B]-B$, respectively. By Proposition 3, $[B]$ is a G_δ-subset of H while B is not. From this it easily follows that $[B]-B$, and hence also X, is not σ-compact.

Example 3. Let $\{X_a\}_{a \in A}$ be a family of topological spaces indexed by a topological space A. In

$$X = A \cup \bigcup_{a \in A} X_a$$

we define a topology as follows: a neighborhood base in X at $x \in X_a$ is given by any neighborhood base at x in X_a, and a neighborhood base in X at $a \in A$ is given by sets $U \cup \bigcup \{X_\beta \colon \beta \in U - (a)\}$, where U is a neighborhood of a in A. It is easy to verify that if A and all X_a's are first countable compact Hausdorff spaces, then so is X. Moreover, if $X_a \neq \emptyset$ for uncountably many $a \in A$, then A is not G_δ in X. It follows that if A is a metrizable compact and $X_a \neq \emptyset$ for uncountably many $a \in A$, then X cannot be imbedded into H.

Finally, we show that, in general, open subsets of H can be quite complicated.

LEMMA 3. *A separable metacompact space is Lindelöf.*

This lemma was stated and proved in [2], Chapter 8, 7.4, p. 176, for paracompact spaces. The proof, however, applies verbatim also to metacompact spaces.

PROPOSITION 4. *An open subset of H is σ-compact if and only if it is metacompact.*

Proof. (a) Let $G \subset H$ be an open metacompact set. Since H is separable (see [3], Chapter 5, Problem M, (c)), so is G. By Lemma 3, G is a countable union of open rectangles, and thus σ-compact, for open rectangles are σ-compact.

(b) Since every σ-compact space is paracompact, the proof in the other direction is trivial.

Since H is not perfectly normal, it indeed contains non-metacompact open subsets.

Remark. Let S be an arbitrary ordered set and let X be the set of all non-decreasing functions from S to I. The set X endowed with the relative product topology is a compact Hausdorff space which is a simple generalization of the Helly space. By the technique employed in the proof of Proposition 3, it can be shown that X is first countable if and only if S is separable in the order topology. This last condition is equivalent to saying that S is homeomorphic to a subset of I.

REFERENCES

[1] P. S. Alexandroff and P. S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam, Afdeeling Natuurkunde, Amsterdam, 1e sectie, 14 (1929), p. 1-96.
[2] J. Dugundji, *Topology*, Boston 1969.
[3] J. L. Kelley, *General topology*, New York 1955.

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Reçu par la Rédaction le 23. 2. 1975