COLLOQUIUM MATHEMATICUM

VOL. XLV 1981 FASC. 1

# *EVERYWHERE DIVERGENT FOURIER SERIES*

BY

T. W. KÖRNER (TRINITY HALL, CAMBRIDGE)

**Introduction.** We write $T = R/2\pi Z$ for the unit circle considered as the real line with $t+2\pi$ identified with $t$. If $\mu$ is a measure on $T$, we write

$$\hat{\mu}(r) = \int_T \exp(-irx)\,d\mu(x) \quad \text{and} \quad S_n(\mu, t) = \sum_{r=-n}^{n} \hat{\mu}(r)\exp(irt).$$

If $f \in L^1(T)$ (i.e. if $f$ is Lebesgue integrable on $T$), we write

$$\hat{f}(r) = \frac{1}{2\pi}\int_T \exp(-irx)\,dx \quad \text{and} \quad S_n(f, t) = \sum_{r=-n}^{n} \hat{f}(r)\exp(irt).$$

Carleson has shown [1] that if $f \in L^2(T)$, then $S_n(f, \cdot) \to f$ almost everywhere with respect to Lebesgue measure. On the other hand, we have the famous theorem of Kolmogorov:

THEOREM A. *There exists an $f \in L^1(T)$ such that $S_n(f, \cdot)$ diverges unboundedly everywhere.*

The purpose of this paper is to give a proof of this and related results. The basic idea remains that of Kolmogorov but the exposition is simplified by using modern notation and a number theoretic result of Kronecker. The idea of using this occurred to Stein and Kahane independently ([7], [3]). We add a further simplification (as compared, say, to the presentation of these ideas by Katznelson [5], Chapter 2) by applying Lemma 1.1. This is the only originality claimed.

**1. Independence.** In this section we lay out the simple number theoretic results that we need.

Definition 1.1. We say that $x_1, x_2, \ldots, x_n \in T$ are *independent* if $\sum_{j=1}^{n} m_j x_j = 0$ with $m_j \in Z$ $(1 \leqslant j \leqslant n)$ only has the solution $m_1 = m_2 = \ldots \ldots = m_n = 0$.

Thus, for example, $1+\pi, 1$ are not independent since $2(1+\pi)+$ $+(-2)\cdot 1 = 0$.

Independence is not preserved under translation but, on the other hand, translation cannot introduce more than one relation.

LEMMA 1.1. *Suppose $x_1, x_2, \ldots, x_n \in T$ are independent and $t \in T$. Then if*

$$\sum_{j=1}^{n} m_j(x_j - t) = 0, \quad \sum_{j=1}^{n} m_j'(x_j - t) = 0 \quad \textit{with } m_j, m_j' \in Z \ (1 \leqslant j \leqslant n),$$

*we can find $k, l \in Z$ not both zero with $km_j = lm_j'$.*

Proof. If $\sum m_j = 0$, then $\sum m_j(x_j - t) = 0$ gives $\sum m_j x_j = 0$, so that $m_1 = m_2 = \ldots = m_n = 0$ and the result follows with $k = 1$, $l = 0$. Thus we may assume that $\sum m_j \neq 0$. Set $k = \sum m_j'$ and $l = \sum m_j$; then

$$\sum_{j=1}^{n} (km_j - lm_j')x_j = k\sum_{j=1}^{n} m_j(x_j - t) - l\sum_{j=1}^{n} m_j'(x_j - t) = 0,$$

and so $km_j - lm_j' = 0$ $(1 \leqslant j \leqslant n)$ as stated.

We recall the basic result on independence due to Kronecker.

THEOREM (Kronecker). *Suppose $x_1, x_2, \ldots, x_n \in T$ are independent. Then given $\lambda_1, \lambda_2, \ldots, \lambda_n \in C$ with $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_n| = 1$ and $\varepsilon > 0$ we can find $m \in Z$ with*

$$|\exp imx_j - \lambda_j| < \varepsilon \quad (1 \leqslant j \leqslant n).$$

Hardy and Wright give a discussion of the theorem together with three different proofs in Chapter XXIII of [2].

As a trivial consequence we have the following corollary:

LEMMA 1.2. *Suppose $x_1, x_2, \ldots, x_n \in T$ are independent. Then given $\tau_1, \tau_2, \ldots, \tau_n$ with $\tau_j$ taking the value $\pm 1$, and given $\varepsilon > 0$ we can find an $m \in Z$ with $m \geqslant 0$ and*

$$\left|\sin\left(m + \frac{1}{2}\right)x_j - \tau_j\right| < \varepsilon \quad (1 \leqslant j \leqslant n).$$

Proof. Take $\lambda_j = \tau_j i \exp(-\tfrac{1}{2}ix_j)$ in the theorem of Kronecker.

Finally, we need to know that we can always find enough independent points. This is easy to show.

LEMMA 1.3. *Suppose $I_1, I_2, \ldots, I_n$ are open intervals in $T$. Then we can find independent points $x_1, x_2, \ldots, x_n$ with $x_j \in I_j$ $(1 \leqslant j \leqslant n)$.*

Proof. Pick rational numbers $q_1, q_2, \ldots, q_n$ such that $2\pi q_j \in I_j$ and a transcendental number $\gamma$. Then if $N$ is a positive integer, the points $x_j = 2\pi(q_j + \gamma^j N^{-1})$ are certainly independent and, provided we take $N$ large enough, it follows that $x_j \in I_j$ $(1 \leqslant j \leqslant n)$.

**2. The basic measure.** Let us choose $n \geqslant 10^6$ and $x_1, x_2, \ldots, x_n \in T$ independent points such that $x_j$ is close to $2\pi j/n$; to be more precise, $|x_j - 2\pi j/n| \leqslant \varepsilon(n)$, where $0 < \varepsilon(n) \leqslant 10^{-4}n^{-1}$ (later we may require $n\varepsilon(n) \to 0$ at some particular speed as $n \to \infty$, but for the proof of Theorem A the reader can take $\varepsilon(n) = 10^{-4}n^{-1}$). If $\delta_{x_j}$ is the Dirac unit point mass at $x_j$, then

$$\hat{\delta}_{x_j}(r) = \exp(-irx_j),$$

and

$$S_m(\delta_{x_j}, t) = \sum_{-m}^{m} \exp\big(ir(t-x_j)\big) = \frac{\sin(m+\frac{1}{2})(t-x_j)}{\sin\frac{1}{2}(t-x_j)},$$

where $[\sin(m+\frac{1}{2})s]/\sin\frac{1}{2}s$ has the value $2m+1$ when $s = 0$.

We define our basic measure to be

$$\mu = n^{-1}\sum_{j=1}^{n} \delta_{x_j}.$$

We note at once that $\|\mu\| = 1$ and that $\mu$ is a positive measure. Further

$$S_m(\mu, t) = n^{-1}\sum_{j=1}^{n} S_m(\delta_{x_j}, t) = \frac{1}{n}\sum_{j=1}^{n} \frac{\sin(m+\frac{1}{2})(t-x_j)}{\sin\frac{1}{2}(t-x_j)}. \tag{1}$$

How large is $S_m(\mu, t)$? Observe that since $|\sin x| \leqslant |x|$ for $|x| \leqslant \pi/2$, we have

$$|\sin\tfrac{1}{2}(t-x_j)|^{-1} \geqslant 2\,|t-x_j|^{-1}.$$

Moreover, since the $x_j$ are more or less uniformly distributed round $T$, so are the $t-x_j$. In particular, if we choose $j(0)$ so that $|t-x_{j(0)}| \leqslant |t-x_j|$ $(1 \leqslant j \leqslant n)$, then

$$|t-x_{j(0)+1}|,\ |t-x_{j(0)-1}| \leqslant \frac{4\pi}{n}, \qquad |t-x_{j(0)+2}|,\ |t-x_{j(0)-2}| \leqslant \frac{6\pi}{n}$$

and, in general,

$$|t-x_{j(0)+r}|,\ |t-x_{j(0)-r}| \leqslant \frac{(2r+2)\pi}{n} \qquad \Big(1 \leqslant r \leqslant \frac{n-2}{2}\Big)$$

(making the obvious convention that $x_{j+n} = x_j$). Thus

$$\frac{1}{n}\sum_{j=1}^{n} \frac{1}{|\sin\frac{1}{2}(t-x_j)|} \geqslant \frac{1}{n}\sum_{1 \leqslant r \leqslant (n-2)/2} \frac{2n}{(2r+2)\pi} \geqslant \frac{1}{\pi}\sum_{2 \leqslant r \leqslant n/2} \frac{1}{r} \geqslant \frac{\log n}{5}.$$

In other words, if we can choose $m$ so that there is no cancellation in our formula (1) for $S_m(\mu, t)$, we can get $|S_m(\mu, t)|$ of the order of $\log n$. Choosing the $x_j$ to be independent ensures that we can always do this.

LEMMA 2.1. *If $\mu$ is chosen as above, then*

$$\sup_{m\geqslant 0}|S_m(\mu,t)|\geqslant 10^{-1}\log n \quad \textit{for all } t\in \mathbf{T}.$$

Proof. There are 4 possible cases to consider the first of which may be regarded as typical.

Case 1. All the $(x_j-t)$ are independent. Then by Lemma 1.2 we can find an $m\geqslant 0$ such that

$$\frac{\sin(m+\frac{1}{2})(t-x_j)}{\sin\frac{1}{2}(t-x_j)}\geqslant\frac{2}{3|\sin\frac{1}{2}(t-x_j)|} \quad \text{for all } j.$$

Case 2. There exist integers $m_1, m_2, \ldots, m_n$ at least 2 of which are non-zero such that $\sum m_j(x_j-t)=0$. Choose $j(0)$ to be a $j$ with $m_j\neq 0$ for which $|x_j-t|$ is the greatest. Then $|x_{j(0)}-t|>\pi/2n$, and so (since $|\sin x|\geqslant 2|x|/\pi$ for $|x|\leqslant\pi/2$) $|\sin\frac{1}{2}(t-x_{j(0)})|^{-1}\leqslant 4n/\pi$. But, by Lemma 1.1, the points $x_j-t$ with $j\neq j(0)$ must be independent, and so we can find an $m\geqslant 0$ such that

$$\frac{\sin(m+\frac{1}{2})(t-x_j)}{\sin\frac{1}{2}(t-x_j)}\geqslant\frac{2}{3|\sin\frac{1}{2}(t-x_j)|} \quad \text{for all } j\neq j(0) \tag{2}$$

and, consequently,

$$S_m(\mu,t)\geqslant\frac{1}{n}\sum_{j\neq j(0)}\frac{2}{3}\frac{1}{|\sin\frac{1}{2}(t-x_j)|}-\frac{1}{n|\sin\frac{1}{2}(t-x_{j(0)})|}$$

$$\geqslant\frac{2}{3n}\sum_{j=1}^{n}\frac{1}{|\sin\frac{1}{2}(t-x_j)|}-\frac{5}{3}\frac{4}{\pi}\geqslant 10^{-1}\log n.$$

Case 3. There exists a $j(0)$ such that $m_0(x_{j(0)}-t)=0$ for some $m_0\geqslant 1$ but $x_{j(0)}-t\neq 0$. Then, by Lemma 1.1, the points $x_j-t$, and so also the points $m_0(x_j-t)$ with $j\neq j(0)$ must be independent. Thus, by the theorem of Kronecker, we can find an integer $q\geqslant 0$ with

$$\frac{\sin(m_0q+\frac{1}{2})(t-x_j)}{\sin\frac{1}{2}(t-x_j)}\geqslant\frac{2}{3|\sin\frac{1}{2}(t-x_j)|} \quad \text{for all } j\neq j(0).$$

Then

$$S_{m_0q}(\mu,t)=\frac{1}{n}\sum_{j\neq j(0)}\frac{2}{3|\sin\frac{1}{2}(t-x_j)|}+\frac{1}{n}\geqslant 10^{-1}\log n.$$

Case 4. There exists a $j(0)$ such that $x_{j(0)}-t=0$. Once again the points $x_j-t$ with $j\neq j(0)$ are independent and we can find an $m$ such that (2) holds. We have

$$S_m(\mu, t) = \frac{1}{n} \sum_{j \neq j(0)} \frac{2}{3\,|\sin\frac{1}{2}(t - x_j)|} + \frac{2m+1}{n} \geqslant 10^{-1}\log n.$$

Since the 4 cases are exhaustive, the lemma is proved.

**3. The basic function.** The contents of this section are routine third year undergraduate analysis. Proofs are given for completeness but it is unlikely that the reader will need to consult them.

LEMMA 3.1. *If $\mu$ is as in Section 2, then we can find an $M$ such that*

$$\sup_{0 \leqslant m \leqslant M} S_m(\mu, t) \geqslant 20^{-1}\log n \quad \textit{for all } t \in \boldsymbol{T}.$$

Proof. For each $t \in \boldsymbol{T}$ we can find an $m(t)$ such that $S_{m(t)}(\mu, t) \geqslant 15^{-1}\log n$. Since $S_{m(t)}(\mu, t)$ is a trigonometric polynomial, so a continuous function, we can find an $\eta(t) > 0$ such that $S_{m(t)}(\mu, s) \geqslant 20^{-1}\log n$ for all $s \in \big(t - \eta(t), t + \eta(t)\big)$. Since $\boldsymbol{T}$ is compact, we can find $t_1, t_2, \ldots, t_r$ such that

$$\bigcup_{k=1}^{r} \big(t_k - \eta(t_k), t_k + \eta(t_k)\big) = \boldsymbol{T}.$$

Setting

$$M = \max_{1 \leqslant k \leqslant r} m(t_k)$$

we have the result.

Now we approximate our measure by a function.

LEMMA 3.2. *There exists an infinitely differentiable positive function $f\colon \boldsymbol{T} \to \boldsymbol{R}$ such that*

(i) $$\operatorname{supp} f \subseteq \bigcup_{j=1}^{n} \left[\frac{2\pi j}{n} - 2\varepsilon(n), \frac{2\pi j}{n} + 2\varepsilon(n)\right],$$

(ii) $$\frac{1}{2\pi} \int_{2\pi j/n - 2\varepsilon(n)}^{2\pi j/n + 2\varepsilon(n)} f(t)\,dt = \frac{1}{n},$$

(iii) $$\sup_{0 \leqslant m \leqslant M} |S_m(f, t)| \geqslant 40^{-1}\log n.$$

Proof. Let $h_r\colon \boldsymbol{T} \to \boldsymbol{R}$ be an infinitely differentiable positive function such that

(i)$_r$ $$\operatorname{supp} h_r \subseteq \left[-\frac{\pi}{r}, \frac{\pi}{r}\right],$$

(ii)$_r$ $$\frac{1}{2\pi} \int_{-\pi/r}^{\pi/r} h_r(t)\,dt = 1.$$

Direct calculation shows that $\hat{h}_r(m) \to 1 = \hat{\delta}_0(m)$ and thus, writing

$$f_r(t) = \mu * h_r(t) = n^{-1} \sum_{j=1}^{n} h_r(t - x_j),$$

we get $\hat{f}_r(m) \to \hat{\mu}(m)$ as $r \to \infty$ for each fixed $m$. Thus $S_m(f_r, \cdot) \to S_m(\mu, \cdot)$ uniformly as $r \to \infty$ for each fixed $m$. Moreover,

$$\operatorname{supp} f_r \subseteq \bigcup_{j=1}^{n} \left[x_j - \frac{\pi}{r}, x_j + \frac{\pi}{r}\right] \quad \text{and} \quad \frac{1}{2\pi} \int_{x_j - \pi/n}^{x_j + \pi/n} f(t)\,dt = \frac{1}{n},$$

so the result follows on taking $r$ large enough and setting $f = f_r$.

Finally, we approximate our function by a trigonometric polynomial.

LEMMA 3.3. *There exists a trigonometric polynomial $P$ given by*

$$P(t) = \sum_{r=-N}^{N} a_r \exp(irt) \qquad (t \in T)$$

*say such that*

(i) $$\frac{1}{2\pi} \int_T |P(t)|\,dt \leqslant 2,$$

(ii) $$\sup_{0 \leqslant m \leqslant N} |S_m(P, t)| \geqslant 40^{-1} \log n.$$

Proof. Since the $f$ of Lemma 3.2 is infinitely differentiable, we may integrate twice by parts to obtain

$$|\hat{f}(m)| \leqslant Am^{-2} \ (m \neq 0), \quad \text{where } A = \frac{1}{2\pi} \int_T |f''(t)|\,dt.$$

In particular, we may find an $N \geqslant M$ such that

$$\sum_{|m|>N} |\hat{f}(m)| \leqslant 1.$$

The Weierstrass $M$ test now tells us that $\sum_{|m|>N} \hat{f}(m) \exp(imt)$ converges uniformly to a continuous function $g$. Using uniform convergence we obtain at once $\hat{g}(m) = \hat{f}(m)$ for $|m| > N$, $\hat{g}(m) = 0$ for $|m| \leqslant N$ and

$$\frac{1}{2\pi} \int_T |g(t)|\,dt \leqslant \sum_{|m|>N} |\hat{f}(m)| \leqslant 1.$$

Set

$$P(t) = \sum_{r=-N}^{N} \hat{f}(r) \exp(irt).$$

Then $P$ is a trigonometric polynomial and $\hat{P}(m) = \hat{f}(m)$ for $|m| \leqslant N$, $\hat{P}(m) = 0$ for $|m| > N$. In particular, $S_m(P, t) = S_m(f, t)$ for $|m| \leqslant N$, so (ii) follows from Lemma 3.2 (iii). Since $(P+g)\hat{}\,(m) = \hat{f}(m)$ for all $m$, it follows from the uniqueness theorem for continuous functions that $P+g = f$. Thus

$$\frac{1}{2\pi}\int_{T} |P(t)|\,dt \leqslant \frac{1}{2\pi}\int_{T} |f(t)|\,dt + \frac{1}{2\pi}\int_{T} |g(t)|\,dt \leqslant 2$$

and (i) holds.

**4. The proof of Kolmogorov's theorem.** We can now prove Kolmogorov's theorem by the condensation of singularities. The proof, in fact, gives slightly more.

THEOREM A.1. *There exists an $f \in L^1(T)$ such that $\hat{f}(r) = 0$ for $r < 0$ and such that $S_n(f, \cdot)$ diverges unboundedly everywhere.*

Proof. By Lemma 3.3 we can find trigonometric polynomials $P_k$ and integers $N(k)$ such that

(i) $$\frac{1}{2\pi}\int_{T} |P_k(t)|\,dt \leqslant 2,$$

(ii) $$\sup_{0 \leqslant m \leqslant N(k)} |S_m(P_k, t)| \geqslant 2^{2k} \quad \text{for all } t \in T,$$

(iii) $$\hat{P}_k(r) = 0 \quad \text{for } |r| > N(k).$$

Set $M(k) = \big(N(k)+1\big) + 2\sum_{q=1}^{k-1}\big(N(q)+1\big)$ and $Q_k(t) = \exp\big(iM(k)t\big)P_k(t)$. Then

(i)′ $$\frac{1}{2\pi}\int_{T} |Q_k(t)|\,dt \leqslant 2,$$

and since $S_m(Q_k, t) = \exp\big(iM(k)t\big)S_{m-M(k)}(P_k, t)$ for $m \geqslant M(k)+N(k)$, we have

(ii)′ $$\sup_{m,n \geqslant 0} |S_m(Q_k, t) - S_n(Q_k, t)| \geqslant \sup_{m \geqslant 0} |S_m(P_k, t)| \geqslant 2^{2k}.$$

Automatically,

(iii)′ $$\hat{Q}_k(r) = 0 \quad \text{for } |r - M(k)| \geqslant N(k)+1.$$

Now observe that since $L^1$ is complete, $\sum_{k=1}^{\infty} 2^{-k}Q_k$ converges in $L^1$ to $f$ say, and $\sum_{k=1}^{\infty} 2^{-k}|Q_k|$ to $\psi$ say. Since

$$|\psi(t)| \geqslant \Big|\sum_{k=1}^{r} 2^{-k}Q_k(t)\Big|$$

almost everywhere, we can use the dominated convergence theorem of Lebesgue to show that

$$\hat{f}(r) = \sum_{k=1}^{\infty} 2^{-k}\hat{Q}_k(r),$$

and so

$$\text{(iv)} \qquad \hat{f}(r) = \begin{cases} 2^{-k}\hat{Q}_k(r) & \text{for } |r-M(k)| \leqslant N(k) \ (k \geqslant 1), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\hat{f}(r) = 0$ for $r < 0$ and

$$\sup_{m,n\geqslant 0} |S_m(f,t) - S_n(f,t)| \geqslant \sup_{M(k)+N(k)\geqslant m,n\geqslant M(k)-N(k)} |S_m(Q_k,t) - S_n(Q_k,t)|$$
$$\geqslant 2^{-k}\sup_{m\geqslant 0} |S_m(P_k,t)| \geqslant 2^k \quad \text{for all } k \geqslant 0.$$

Thus $S_m(f,t)$ diverges unboundedly for all $t \in \boldsymbol{T}$.

**5. Further remarks on Kolmogorov's theorem.** This section contains remarks of more specialized interest and should be omitted by the general reader.

Remark 1. Minor variations of the condensation method used in Section 4 yield minor variations of the theorem.

THEOREM A.2. *There exists a real function $f \in L^1(\boldsymbol{T})$ such that $S_n(f,\cdot)$ diverges everywhere.*

Proof. Checking through the work of Section 3 we see that the $P$ of Lemma 3.3 is, in fact, real. Thus proceeding inductively we can find real trigonometric polynomials $P_k$ and a strictly increasing sequence of positive integers $N(k)$ such that (writing $N(0) = 1$)

$$\text{(i)} \qquad \frac{1}{2\pi}\int_{\boldsymbol{T}} |P_k(t)| \leqslant 2,$$

$$\text{(ii)} \qquad \sup_{|m|\leqslant N(k)} |S_m(P_k,t)|$$
$$\geqslant 2^{4k+1}\big(N(k-1)+1\big)\Big(1+\sum_{l=1}^{k-1}\sup_{|m|\leqslant N(l)}\sup_{t\in \boldsymbol{T}} |S_m(P_l,t)|\Big),$$

$$\text{(iii)} \qquad \hat{P}_k(r) = 0 \quad \text{for } |r| > N(k).$$

Note that condition (i) implies $|\hat{P}_k(m)| \leqslant 2$, and so

$$|S_m(P_k,t)| \leqslant 2m+1 \leqslant 2N(k-1)+1 \quad \text{for } 0 \leqslant m \leqslant N(k-1).$$

Simple computation shows that $\sum_{k=1}^{\infty} 2^{-2k-1}\big(N(k-1)+1\big)^{-1}P_k$ converges in $L^1$ to a real function $f$ such that

$$\limsup_{m\to\infty} |S_m(f, t)| = \infty \quad \text{for all } t \in \boldsymbol{T}.$$

Remark 2. Our calculations do not tell us how fast $S_n(f, \cdot)$ diverges. Indeed, if the $x_1, x_2, \ldots, x_n$ are chosen arbitrarily, then no bound can be put on the $N$ of Lemma 3.3. However, if we take $x_j = 2\pi(100^{-j}+j/n)$ $(1 \leqslant j \leqslant n)$ (though now the $x_j$ are not, strictly speaking, independent but only "almost independent"), we can show using the quantitative version of Kronecker's theorem given in Appendix V of [4] (or barehanded methods) and the arguments of Section 3 that the following is true:

LEMMA 2.1′. *With the $x_j$ chosen as in the paragraph above we have*

$$\sup\{|S_m(\mu, t)|:\ 0 \leqslant m \leqslant 100^n\} \geqslant 10^{-1}\log n \quad \textit{for all } t \in \boldsymbol{T}.$$

Next examining the approximations to the function $\delta$ we get the following result:

LEMMA 3.2′. *There exists an infinitely differentiable positive function $f\colon \boldsymbol{T} \to \boldsymbol{R}$ such that*

(i) $$0 \leqslant f(t) \leqslant 100^{n+2},$$

(ii) $$\int_{\boldsymbol{T}} f(t)\,dt = 1,$$

(iii) $$\sup\{|S_m(f, t)|:\ 0 \leqslant m \leqslant 100^n\} \geqslant 40^{-1}\log n.$$

This can be trivially rewritten as

LEMMA 3.2″. *There exists an infinitely differentiable positive function $f\colon \boldsymbol{T} \to \boldsymbol{R}$ such that*

(i) $$0 \leqslant f(t) \leqslant N,$$

(ii) $$\int_{\boldsymbol{T}} f(t)\,dt = 1,$$

(iii) $$\sup\{|S_m(f, t)|:\ 0 \leqslant m \leqslant N\} \geqslant 10^{-6}\log\log N.$$

The argument of Section 4 and Remark 1 of this section give

LEMMA 5.1. *If $\psi(n) = O(\log\log n)$ as $n \to \infty$, then we can find an $f \in L^1(\boldsymbol{T})$ (which may be taken to be real or to have $\hat{f}(r) = 0$ for $r < 0$) such that*

$$\limsup_{m\to\infty} \frac{|S_m(f, t)|}{\psi(m)} = \infty \quad \textit{for all } t \in \boldsymbol{T}.$$

LEMMA 5.2. *If* $\psi(x) = O(\log\log x)$ *as* $x \to \infty$, *then we can find an* $f \in L^1(T)$ (*which may be taken to be real or to have* $\hat{f}(r) = 0$ *for* $r < 0$) *such that*

$$\int_T |f(t)|\,\psi(|f(t)|)\,dt < \infty$$

*but*

$$\limsup_{m\to\infty} |S_m(f, t)| = \infty \quad \text{for all } t \in T.$$

We thus recover the results of Tandori [8].

It is known (by a remark of Carleson expanded by Sjölin [6]) that $\psi(n) = O(\log\log n)$ cannot be replaced by $\psi(n) = O(\log n \log\log n)$. Better results are claimed by M. and S. Izumi but these are still controversial.

Remark 3. By direct construction of suitable $x_j$ with $|x_j - 2\pi j/n| \leqslant 10^{-4}/n$ $(1 \leqslant j \leqslant n)$ we can prove the following result:

LEMMA 5.3. *Suppose* $m(1), m(2), \ldots$ *is a sequence of positive integers with* $m(r) \to \infty$. *Then given an integer* $p \geqslant 1$ *we can find* $x_1, x_2, \ldots, x_n$ *and a collection of positive integers* $s(1), s(2), \ldots, s(nP)$ *with* $m(s(r+1)) > 10m(s(r))$ *such that* $\mu = n^{-1}\sum_{j=1}^{n}\delta_{x_j}$ *has the following property:*

*If* $(j-1)p+1 \leqslant s \leqslant jp$, *then*

$$|S_{m(s(r))}(\mu, t)| \geqslant 40^{-1}\log n$$

*whenever* $t \in [x_{j-1}, x_j]$, $|t - 2\pi u/m(s(r))| \leqslant \big(10^4 m(s(r))\big)^{-1}$ *for some integer* $u$. (*Here, as usual,* $x_0 = x_n$.)

Repeating with minor modifications the construction used in Remark 1 of this section we get a version of another theorem of Kolmogorov.

THEOREM 5.1. *Given a sequence of positive integers* $m(r) \to \infty$ *we can find a real function* $f \in L^1(T)$ *such that*

$$\limsup_{r\to\infty} S_{m(r)}(f, t) = \infty \quad \text{for almost all } t \in T.$$

It can be shown that if $m(r+1) \geqslant \lambda m(r)$ for all $r \geqslant 1$ and some $\lambda > 1$, then whenever $f \in L^1(T)$ and $\hat{f}(k) = 0$ for all $k < 0$ it follows that $S_{m(r)}(f, t)$ converges almost everywhere ([9], Chap. XV, § 4). It is also not very difficult to show that, provided $m(r) \to \infty$ sufficiently fast, then if $f \in L^1(T)$ is a real function, we can find an $x \in T$ such that $S_{m(r)}(f, x)$ converges as $r \to \infty$.

**6. Bounded divergence. A theorem of Marcinkiewicz.** Let us look again at the basic function $f$ of Lemma 3.2.

LEMMA 6.1. *If* $(\log n)^{-1} \leqslant n\varepsilon(n)$ *and* $f\colon T \to R$ *is an infinitely differentiable positive function satisfying conditions* (i) *and* (ii) *of Lemma* 3.2, *then*

(iv) $|S_m(f,t)| \leqslant 40\log n$

*provided only that* $t \notin \bigcup_{j=1}^{n}\left[\frac{2\pi j}{n}-4\varepsilon(n),\ \frac{2\pi j}{n}+4\varepsilon(n)\right]$.

Proof. We have

$$|S_m(f,t)| = \left|\frac{1}{2\pi}\int_T \sum_{r=-m}^{m} \exp\big(ir(t-x)\big)f(x)\,dx\right|$$

$$= \frac{1}{2\pi}\left|\int_T \frac{\sin(m+\frac{1}{2})(t-x)}{\sin\frac{1}{2}(t-x)}f(x)\,dx\right|$$

$$= \sum_{j=1}^{n}\left|\int_{2\pi j/n-2\varepsilon(n)}^{2\pi j/n+2\varepsilon(n)} \frac{\sin(m+\frac{1}{2})(t-x)}{\sin\frac{1}{2}(t-x)}f(x)\,dx\right|$$

$$\leqslant \sum_{j=1}^{n}\frac{1}{n}\sup\left\{\left|\frac{\sin(m+\frac{1}{2})(t-x)}{\sin\frac{1}{2}(t-x)}\right| : \left|x-\frac{2\pi j}{n}\right| \leqslant 2\varepsilon(n)\right\}$$

$$\leqslant \sum_{j=1}^{n}\frac{1}{n}\sup\left\{\frac{\pi}{2}\frac{2}{t-x} : \left|x-\frac{2\pi j}{n}\right| \leqslant 2\varepsilon(n)\right\}$$

$$\leqslant \frac{\pi}{n}\left(2n\log n + 3\sum_{0\leqslant r\leqslant n/2}\frac{n}{2r+1}\right) \leqslant 40\log n$$

bearing in mind, first, that $|\sin t| \geqslant 2t/\pi$ for $|t| \leqslant \pi/2$ and, second, that if we choose $j(0)$ so that $|t-2\pi j(0)/n| \leqslant |t-2\pi j/n|$ for $1 \leqslant j \leqslant n$, then

$$\left|t-\frac{2\pi\big(j(0)+1\big)}{n}\right|,\ \left|t-\frac{2\pi\big(j(0)-1\big)}{n}\right| \geqslant \frac{\pi}{n},$$

$$\left|t-\frac{2\pi\big(j(0)+2\big)}{n}\right|,\ \left|t-\frac{2\pi\big(j(0)-2\big)}{n}\right| \geqslant \frac{3\pi}{n}$$

and, in general,

$$\left|t-\frac{2\pi\big(j(0)+r\big)}{n}\right|,\ \left|t-\frac{2\pi\big(j(0)-r\big)}{n}\right| \geqslant \frac{(2r-1)\pi}{n}.$$

Finally, we note that if $|x-2\pi j(0)/n| \leqslant 2\varepsilon(n)$ and $|t-2\pi j(0)/n| \geqslant 4\varepsilon(n)$, then

$$|x-t| \geqslant 2\varepsilon(n) \geqslant 2(n\log n)^{-1},$$

which allows us to complete our estimate.

We can now sharpen Lemma 3.3 without further work.

LEMMA 6.2. *There exist a trigonometric polynomial $P$ and a closed set $E$ such that*

(i) $$\frac{1}{2\pi}\int_{T}|P(t)|\,dt \leqslant 2, \quad |P(t)| \leqslant 1 \quad \textit{for } t \notin E,$$

(ii) $$\sup_{m>0}|S_m(P,t)| \geqslant 40^{-1}\log n,$$

(iii) $$\operatorname{meas}(E) \leqslant 8n^{-1}\log n,$$

(iv) $$50\log n \geqslant |S_m(P,t)| \quad \textit{for all } t \notin E,\ m \in \mathbf{Z}.$$

Using this we get the remarkable theorem of Marcinkiewicz.

THEOREM B.1. *There exists an $f \in L^1(T)$ such that $\hat{f}(r) = 0$ for $r < 0$ and such that $S_r(f, \cdot)$ diverges boundedly almost everywhere and diverges everywhere.*

Proof. By Lemma 6.2 we can find closed sets $E_k$, trigonometric polynomials $P_k$ and integers $N(k)$ such that

(i) $$\frac{1}{2\pi}\int_{T}|P_k(t)|\,dt \leqslant 2, \quad |P_k(t)| \leqslant 1 \quad \text{for } t \notin E_k,$$

(ii) $$\sup_{0<m\leqslant N(k)}|S_m(P_k,t)| \geqslant 40^{-1}\cdot 2^k,$$

(iii) $$\hat{P}_k(r) = 0 \quad \text{for } r > N(k),$$

(iv) $$\operatorname{meas}(E_k) \leqslant 2^{-k},$$

(v) $$50\cdot 2^k \geqslant |S_m(P_k,t)| \quad \text{for all } t \notin E_k,\ m \in \mathbf{Z}.$$

Set

$$M(k) = \big(N(k)+1\big)+2\sum_{q=1}^{k-1}\big(N(q)+1\big) \quad \text{and} \quad Q_k(t) = \exp\big(iM(k)t\big)P_k(t).$$

As in the proof of Kolmogorov's theorem in Section 4 we see that $\sum_{k=1}^{\infty} 2^{-k}Q_k$ converges in $L^1$ to a function $f$ which satisfies (iv) in Section 4.

Now suppose $t \notin \sum_{k=p}^{\infty} E_k$. Then if $k \geqslant p$, we know from (i) and from (iv) in Section 4 that

$$|S_{M(k)+N(k)}(f,t) - S_{M(k)-N(k)}(f,t)| = 2^{-k}|Q_k(t)| = 2^{-k}|P_k(t)| \leqslant 2^{-k},$$

whilst if $|r-M(k)| \leqslant N(k)$, we infer from (i) and (v) by a similar argument that

$$|S_r(f, t) - S_{M(k)-N(k)}(f, t)| = \left|2^{-k} \sum_{s=M(k)-N(k)}^{r} \hat{Q}_k(s)\exp(ist)\right|$$

$$= 2^{-k}\left| \sum_{s=-N(k)}^{r-M(k)} \hat{P}_k(s)\exp(ist)\right|$$

$$\leqslant 2^{-k}\cdot 2 \sup_{0\leqslant m\leqslant N(k)} |S_m(P_k, t)| \leqslant 100.$$

Thus

$$|S_r(f, t) - S_{M(p)-N(p)}(f, t)| \leqslant 200 + \sum_{k=p}^{\infty} 2^{-k} \leqslant 201,$$

$$\limsup_{r\to\infty} |S_r(f, t)| < \infty.$$

We have thus shown that if $t \notin E = \bigcap_{p=1}^{\infty} \bigcup_{k=p}^{\infty} E_k$, then

$$\limsup_{r\to\infty} |S_r(f, t)| < \infty.$$

But $E \subseteq \bigcup_{k=p}^{\infty} E_k$, so by (iv)

$$\operatorname{meas}(E) \leqslant \sum_{k=p}^{\infty} \operatorname{meas}(E_k) \leqslant 2^{-p+1} \quad \text{for all } p \geqslant 1,$$

i.e. $\operatorname{meas}(E) = 0$.

Finally, just as in the proof of Kolmogorov's theorem (Theorem A1), we note that $\hat{f}(r) = 0$ for $r < 0$ and that

$$\sup_{m,n\geqslant q} |S_m(f, t) - S_n(f, t)| \geqslant \sup_{M(k)+N(k)\geqslant m,n\geqslant M(k)-N(k)} 2^{-k} |S_m(Q_k, t) - S_n(Q_k, t)|$$

$$\geqslant 2^{-k} \sup_{m\geqslant 0} |S_m(P_k, t)| \geqslant 40^{-1}$$

for all $k$ with $M(k) - N(k) \geqslant q$, so that $\limsup_{m,n\to\infty} |S_m(f, t) - S_n(f, t)| \geqslant 40^{-1}$ for all $t \in T$ and $S_m(f, \cdot)$ diverges everywhere.

**7. Further remarks on Marcinkiewicz's theorem.** This section, like Section 5, is devoted to remarks of more specialized interest and should be omitted by the general reader.

Remark 1. Methods identical with those of Section 5 produce results corresponding to Theorem A.2, Lemma 5.2 and Theorem 5.1.

THEOREM B.2. *There exists a real function $f \in L^1(T)$ such that $S_n(f, \cdot)$ diverges boundedly almost everywhere and diverges everywhere.*

LEMMA 7.1. *If $\psi$: $\boldsymbol{R} \to \boldsymbol{R}$ is a continuous positive function with $\psi(x) = O(\log\log x)$ as $x \to \infty$, then we can find an $f \in L^1(\boldsymbol{T})$ (which may be taken to be real or to have $\hat{f}(r) = 0$ for $r < 0$) such that*

$$\int_{\boldsymbol{T}} |f(t)|\psi(|f(t)|)dt < \infty,$$

*but $S_n(f, \cdot)$ diverges boundedly almost everywhere and diverges everywhere.*

LEMMA 7.2. *Given a sequence of positive integers $m(r) \to \infty$ we can find a real function $f \in L^1(\boldsymbol{T})$ such that $S_{m(r)}(f, t)$ diverges for almost all $t$ but*

$$\sup_{m \geqslant 0} |S_m(f, t)| < \infty \quad \textit{for almost all } t.$$

Remark 2. The estimates of Lemma 6.1 and Lemma 3.2 (the latter depending basically on the estimate of Lemma 2.1) can clearly be refined without any difficulty.

LEMMA 7.3. *Suppose we are given $\varepsilon(n), \varepsilon'(n) > 0$ with $n\log n\varepsilon(n) \leqslant 1$, $n\log n\varepsilon'(n) \to \infty$. Then we can find positive infinitely differentiable functions $f_n$: $\boldsymbol{T} \to \boldsymbol{R}$ and positive numbers $\delta(n) \to 0$ such that*

(i) $$\operatorname{supp} f_n \subseteq \bigcup_{j=1}^{n} \left[\frac{2\pi j}{n} - 2\varepsilon(n), \frac{2\pi j}{n} + 2\varepsilon(n)\right],$$

(ii) $$\frac{1}{2\pi} \int_{2\pi j/n - 2\varepsilon(n)}^{2\pi j/n + 2\varepsilon(n)} f_n(t)dt = 1 \quad (j = 1, 2, \ldots, n),$$

(iii) $$\sup_{m \geqslant 0} |S_m(f_n, t)| \geqslant (1 - \delta(n)) \int_{1/n}^{\pi} \frac{1}{\sin \tau/2} d\tau \quad \textit{for all } t \in \boldsymbol{T},$$

(iv) $$\sup_{m \geqslant 0} |S_m(f_n, t)| \leqslant (1 + \delta(n)) \int_{1/n}^{\pi} \frac{1}{\sin \tau/2} d\tau$$

$$\textit{for all } t \notin \bigcup_{j=1}^{n} \left[\frac{2\pi j}{n} - 2\varepsilon'(n), \frac{2\pi j}{n} + 2\varepsilon'(n)\right].$$

From this version of Lemma 6.1 it is easy to get, by the methods of Section 6, the following quantitative version of Theorem B.1:

LEMMA 7.4. *Let $h$: $\boldsymbol{R} \to \boldsymbol{R}$ be an increasing continuous function with $h(0) = 0$ and $h(t) = O(t\log t^{-1})$ as $t \to 0+$. Then we can find a set $E$ of Hausdorff $h$-measure 0 and a function $f \in L^1(\boldsymbol{T})$ (which may be taken to be real or to have $\hat{f}(r) = 0$ for $r < 0$) such that*

$$\limsup_{m,n \to \infty} |S_m(f, t) - S_n(f, t)| = 1 \quad \textit{for all } t \notin E$$

*and $S_n(f, \cdot)$ diverges everywhere.*

In the next and final section we will give an argument of Marcinkiewicz which shows that if $f \in L^1(\boldsymbol{T})$ diverges almost everywhere, then it must diverge unboundedly on a set $E'$ which is everywhere dense. It might, therefore, be interesting to investigate the Hausdorff dimension of this residual set $E'$.

**8. A converse theorem of Marcinkiewicz.** We conclude this exposition by giving a slightly expanded version of Zygmund's account of Marcinkiewicz's elegant converse result.

THEOREM C. *If $f \in L^1$ and $S_n(f, \cdot)$ diverges almost everywhere (with respect to Lebesgue measure), then $S_n(f, \cdot)$ diverges unboundedly on a dense subset of $\boldsymbol{T}$.*

Thus we cannot improve Theorem B.1.

We require 4 results the first 3 of which are easy to prove but the last — the well-known theorem of Carleson — is still very difficult.

LEMMA 8.1. *If $f_1, f_2, \ldots$ are continuous functions on $\boldsymbol{T}$ and we can find an open interval $I$ such that $\sup_j |f_j(x)| < \infty$ for each $x \in I$, then we can find an open interval $J \subset I$ and a $K > 0$ such that $|f_j(x)| < K$ for all $j \geqslant 1$, $x \in J$.*

Proof. By Baire's category theorem, at least one of the sets $E_n = \{x \in I : |f_j(x)| \leqslant n \text{ for all } j \geqslant 1\}$ must have a non-empty interior.

LEMMA 8.2. *If $f \in L^1$ and $\limsup_{n\to\infty} |S_n(f, x)| \leqslant K$ for almost all $x \in I$, $I$ being an interval, then $|f(x)| \leqslant K$ for almost all $x \in I$.*

Proof. Since $f \in L^1$, we have

$$\sigma_n f = \frac{S_0 f + S_1 f + \ldots + S_n f}{n+1} \to f$$

almost everywhere ([5], p. 20, or [9], p. 90). The result follows.

LEMMA 8.3 (principle of localization). *If $f, g \in L^1$, $\varepsilon > 0$ and $f(t) = g(t)$ for almost all $t \in [x-\varepsilon, x+\varepsilon]$, then $|S_n(f, x) - S_n(g, x)| \to 0$.*

For the proof see [9], p. 52.

THEOREM (Carleson). *If $f \in L^2$ (and so, in particular, if $f \in L^\infty$), then $S_n(f, \cdot)$ converges almost everywhere to $f$.*

For the proof see [1].

The proof of Theorem C is obtained by a simple application of these results.

Proof of Theorem C. Suppose that $f \in L^1$ and that $S_n(f, \cdot)$ does not diverge unboundedly on a dense subset of $T$, i.e. that we can find an open interval $I$ such that

$$\sup_n |S_n(f, x)| < \infty \quad \text{for each } x \in I.$$

By Lemma 8.1 we can find a $K > 0$ and an open interval $J$ such that $|S_n(f, x)| \leqslant K$ for each $n \geqslant 1$ and $x \in J$.

Define the characteristic function $\xi_J$ of $J$ by $\xi_J(t) = 1$ if $t \in J$, $\xi_J(t) = 0$ otherwise, and set $g = \xi_J f$. By the Riemann localization principle (Lemma 8.3), $S_n(g, t) \to 0$ for all $t$ not in the closure of $J$ and

$$\limsup_{n\to\infty} |S_n(g, t)| = \limsup_{n\to\infty} |S_n(f, t)| \leqslant K \quad \text{for all } t \in J.$$

But $g \in L^1$ so, by Lemma 8.2, $|g(t)| \leqslant K$ for almost all $t \in T$. Thus $g \in L^\infty$ and so, by Carleson's theorem, $S_n(g, t) \to g$ almost everywhere. Using the localization principle once again, we obtain $S_n(f, t) \to g(t) = f(t)$ for almost all $t \in J$. Thus $f$ cannot diverge almost everywhere and the theorem is proved.

## *REFERENCES*


[1] L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Mathematica 116 (1966), p. 135-157.

[2] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford 1938.

[3] J.-P. Kahane, *Sur la divergence presque sûre presque partout de certaines séries de Fourier aléatoires*, Annales Universitatis Scientiarum Budapestinensis, Sectio Mathematica, 3-4 (1960-1961), p. 101-108.

[4] — et R. Salem, *Ensembles parfaits et séries trigonométriques*, Hermann, 1963.

[5] Y. Katznelson, *An introduction to harmonic analysis*, J. Wiley, 1968.

[6] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh-Fourier series*, Arkiv för Matematik 7 (1969), p. 551-570.

[7] E. M. Stein, *On limits of sequences of operators*, Annals of Mathematics 74 (1961), p. 140-170.

[8] K. Tandori, *Ein Divergenzsatz für Fourierreihen*, Acta Scientiarum Mathematicarum (Szeged) 30 (1969), p. 43-48.

[9] A. Zygmund, *Trigonometric series*, Vol. 2, 2nd edition, Cambridge 1959.



*Reçu par la Rédaction le 5. 1. 1979;*
*en version modifiée le 20. 8. 1979*