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INDIVIDUAL FACTORIZATION IN BANACH MODULES

BY

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Let A be a Banach algebra. The Cohen factorization theorem [5] states that A factorizes (i.e., every element of A can be written as a product) if A has a bounded left approximate identity. One cannot hope for such a factorization theorem under the weaker assumption of an unbounded approximate identity $\{e_\beta\}$, even if the norms $\|e_\beta\|$ tend to infinity at a prescribed rate. This can be seen for instance by looking at the Banach algebra given in Remark a) after Theorem 1, since for any net $\{d_\beta\}$ in $[1, \infty)$ with $d_\beta \to \infty$, it contains an approximate identity $\{e_\beta\}$ with $\|e_\beta\| = d_\beta$. Despite of this fact that for unbounded approximate identities no global factorization result is possible, one still can obtain "local" results, i.e., factorization for specific elements. This is what we shall consider. The main result is Theorem 4. Roughly speaking, it states that the elements of certain approximation spaces in the Banach module X can be written as products.

A simple proof of Cohen's factorization theorem is added in the Appendix.

Throughout this note, A denotes a real or complex Banach algebra, X a left Banach-A-module with module constant $\varkappa \geqslant 1$: $\|ax\|_X \leqslant \varkappa \|a\| \|x\|_X$. The norm closure of a set B is denoted by $\bar{B}$.

THEOREM 1. *Let $x \in X$ be such that*:

(1) *There are $\alpha < 1$ and $K > 0$ such that for every $\varepsilon > 0$ there is $e \in A$ with $\|e\| \leqslant K \cdot \varepsilon^{-\alpha}$ and $\|ex - x\|_X < \varepsilon$.*

(2) *If $\varepsilon_i \to 0$ is a strictly decreasing sequence of positive numbers, there is a sequence $\{e_i\}$ in A where e_i corresponds to ε_i according to* (1) *(with α and K fixed) and $e_{i+1} e_i = e_i e_{i+1} = e_i$.*

Then for every $\varepsilon > 0$ there are $a \in A$ and $y \in \overline{Ax} \subset X$ such that

(i) $x = ay$;

(ii) $\|y - x\|_X < \varepsilon$.

(It actually suffices to satisfy the assumptions for just one suitable sequence $\{\varepsilon_i\}$ and a corresponding sequence $\{e_i\}$, that is: (1) should hold for ε_i and e_i, and one should have $e_{i+1}e_i = e_i e_{i+1} = e_i$. One may for instance consider $\varepsilon_n = n^{-\beta}$ where $1 < \beta < \alpha^{-1}$.)

Remark. a) The above theorem is best possible in the following sense: If in (1) the norm estimate $K\varepsilon^{-\alpha}$ with $\alpha < 1$ is weakened to $K\varepsilon^{-1}$, the theorem fails. We see this from the following example. Let A be the space of all bounded functions f: $[1, \infty) \to \mathbf{R}$ such that $t \mapsto tf(t)$ vanishes at infinity. With pointwise operations and norm $\|f\| = \sup\{|tf(t)| \mid t \in [1, \infty)\}$, A is a Banach algebra. Consider the function $g(t) = t^{-2}$. We have $g \in A$. Let $\varepsilon > 0$ and e be the characteristic function of $[1, 1/\varepsilon]$. We have $\|eg - g\| = \varepsilon$ and $\|e\| = 1/\varepsilon$. Condition (2) of the theorem is also clearly satisfied, so g satisfies the assumptions of the theorem with $\alpha = 1$. But it is impossible to write g as a product $h \cdot k$ with $h, k \in A$, since $1 = t^{-2} \cdot t^2$ does not vanish at infinity.

b) One cannot hope in general to obtain all factorable elements of a given Banach-A-module X by the above theorem, even if $X = A$. If $z \in A$ is factorizable, it need not be contained in the closure of Az (see [12]).

A proof of Theorem 1 can be obtained by refining and modifying to some extent the proof of Cohen's theorem given in the Appendix.

THEOREM 2. *Let X be a Banach module over a Banach algebra A. Let $x \in X$ be such that there exist a sequence $\{e_n\}_{n \geq 1}$ in A, some $k \in \mathbf{N} \cup \{0\}$, $\alpha \geq 0$, $\beta > 1$, and positive constants K_1, K_2 such that*

(i) $e_n e_j = e_j e_n = e_n$ *for* $j \geq n+k$;

(ii) $\|e_n x - x\|_X \leq K_1 2^{-n\alpha} n^{-\beta}$ *for* $n \geq 1$;

(iii) $\|e_n\|_A \leq K_2 2^{n\alpha} n^{-\beta}$ *for* $n \geq 1$.

Then there exist sequences $\{a_j\}_{j=1}^{2k+1}$ in A and $\{y_j\}_{j=1}^{2k+1}$ in X such that $x = \sum_{j=1}^{2k+1} a_j y_j$. In particular, one has $x = ay$ in the case $k = 0$.

Proof. Set $e_m = 0$ for $m \leq 0$ and $d_n = e_n - e_{n-1}$ for $n \in \mathbf{Z}$. Let $d_n^* = \sum_{j=n-k}^{n+k} d_j$ (i.e., $d_n^* = e_{n+k} - e_{n-k-1}$).

As a consequence of (i) we have

$$(3) \qquad \begin{aligned} d_n d_n^* &= d_n^* d_n = d_n && \text{for all } n, \\ d_n d_j^* &= d_j^* d_n = 0 && \text{for } |n-j| \geq 2k+1. \end{aligned}$$

From (ii) we obtain

$$(4) \qquad \|d_{n+1} x\|_X \leq 2K_1 2^{-n\alpha} n^{-\beta},$$

so $\sum d_n x$ converges absolutely and

$$(5) \qquad x = \lim e_n x = \sum_{n=1}^{\infty} d_n x.$$

According to (iii) we have for $n > k+1$

$$\|d_n^*\|_A \leqslant K_2 2^{(n+k)\alpha}(n+k)^{-\beta} + K_2 2^{(n-k-1)\alpha}(n-k-1)^{-\beta} \leqslant K_3 2^{n\alpha}(n-k-1)^{-\beta}. \tag{6}$$

Define now for $j = 1, \ldots, 2k+1$

$$I_j = \{j+(2k+1)i \mid i = 0, 1, 2, \ldots\}, \quad x_j = \sum_{n \in I_j} d_n x. \tag{7}$$

In view of (3) we can write

$$x_j = \sum_{n \in I_j} (2^{-n\alpha} d_n^*)(2^{n\alpha} d_n x) = \Big(\sum_{n \in I_j} 2^{-n\alpha} d_n^*\Big)\Big(\sum_{n \in I_j} 2^{n\alpha} d_n x\Big) = a_j y_j. \tag{8}$$

The two series in brackets converge absolutely because of (4) and (6). So we obtain

$$x = \sum_{j=1}^{2k+1} x_j = \sum_{j=1}^{2k+1} a_j y_j.$$

Remark. f) Since the coefficients of d_n^* and $d_n x$ in (8) can be changed for finitely many n (subject to the condition that, for fixed n, the product stays 1) we can achieve $\sum_1^{2k+1} y_j$ to be arbitrarily close to x. In particular, if $k = 0$ we can choose y arbitrarily close to x. By doing so, the norm of the first factor may possibly be increased.

g) Suppose $k = 0$. As in the case of Cohen's factorization theorem it is possible to factor large sets $M \subset X$ over the same element $a \in A$: Whenever all $x \in M$ satisfy the required estimates for the same sequence $\{e_n\}_{n \geqslant 1}$ in A, the corresponding first factor can be supposed to be the same.

h) Suppose $k > 0$. If we consider the subsequence $e'_n = e_{nk}$, then (ii) and (iii) in the above theorem are still satisfied (with α replaced by $k\alpha$) whereas (i) is improved to $e'_n e'_{n+1} = e'_{n+1} e'_n = e'_n$ (i.e. we have $k = 1$ for the new sequence). Hence we obtain

COROLLARY 3. *Under the hypotheses of Theorem 2 the element x can be written as a sum of three products: $x = a_1 y_1 + a_2 y_2 + a_3 y_3$ (independent of $k \geqslant 1$).*

Looking at the proof of Theorem 2 we see that it works in more general cases. Roughly speaking we need that the product $\|e_n\|_A \|e_n x - x\|_X$ goes to zero sufficiently fast. For instance we have the following theorem:

THEOREM 4. *Let X be a Banach module over a Banach algebra A and let $x \in X$ be such that there are a sequence $\{e_n\}_{n \geqslant 1}$ in A, a positive sequence*

$c = \{c_n\}_{n\geq 1}$ in l^1, a strictly positive sequence $r = \{r_n\}_{n\geq 1}$ and constants $K \in \mathbf{R}^+$, $k \in \mathbf{N} \cup \{0\}$ satisfying the following conditions:

(i) $e_j e_n = e_n e_j = e_n$ *for* $j \geq n+k$;

(ii) $\|e_n x - x\|_X \leq r_n^{-1} c_n^2$ *for* $n \geq 1$;

(iii) $\|e_n\|_A \leq r_n$ *for* $n \geq 1$;

(iv) $K^{-1} r_n \leq r_{n+1} \leq K r_n$ *and* $K^{-1} c_n \leq c_{n+1} \leq K c_n$ *for* $n \geq 1$.

Then there exist elements $a_i \in A$ and $y_i \in X$ such that $x = \sum_{i=1}^{2k+1} a_i y_i$.

In particular, $x = ay$ in the case $k = 0$.

Proof. We may assume $\{c_n\}$ to be strictly positive. Repeat the proof of Theorem 2. The estimates (4) and (6) are now replaced by

$$\|d_n x\|_X \leq K_1 r_n^{-1} c_n^2 \tag{4'}$$

and

$$\|d_n^*\|_A \leq K_2 r_n. \tag{6'}$$

We have

$$x_j = \Big(\sum_{n\in I_j} r_n^{-1} c_n d_n^*\Big)\Big(\sum_{n\in I_j} r_n c_n^{-1} d_n x\Big),$$

the two series in brackets converging absolutely.

We come now to the applications of the individual factorization results. The first concerns Banach spaces of measurable functions with an algebra A acting by pointwise multiplication. More precisely, let (Ω, Σ, μ) be a σ-finite measure space, and let $(X, \|\ \|_X)$ be a solid BF-space of (classes of) locally integrable functions on Ω. Recall that $(X, \|\ \|_X)$ is called *solid* whenever for $f \in X$ and any locally integrable function g on Ω satisfying $|g(z)| \leq |f(z)|$ μ-almost everywhere it follows that g belongs to X and satisfies $\|g\|_X \leq \|f\|_X$ (i.e., X is a Banach module over $L^\infty(\Omega, \Sigma, \mu)$ with respect to pointwise multiplication). Under these conditions we have

THEOREM 5. *Let $f \in X$ and $p \geq 1$ be given. Suppose there exist $K > 0$ and $d > 1/p+1$ such that for any $n \in N$ one can find $g_n \in X$ satisfying $\mu(\operatorname{supp} g_n) \leq 2^n$ and $\|f - g_n\|_X \leq K 2^{-n/p} n^{-d}$. Then there exist $h \in L^p(\Omega, \Sigma, \mu)$ and $g \in X$ such that $f = hg$. In the case $X = L^q(\Omega, \Sigma, \mu)$ it is sufficient to suppose $d > 1/q + 1/p$.*

Proof. We choose $A := L^p \cap L^\infty(\Omega, \Sigma, \mu)$, with its natural norm $\|\ \|_p + \|\ \|_\infty$ as Banach algebra, operating on X by pointwise multiplication. Let e_n be the characteristic function of $\bigcup_{k=1}^{n} \operatorname{supp} g_k$. Then one has $\|e_n\|_A \leq 1 + 2^{(n+1)/p} \leq 4 \cdot 2^{n/p}$, and using the identity $g_n = e_n g_n$

$$\begin{aligned}\|f - fe_n\|_X &\leq \|f - g_n\|_X + \|e_n(g_n - f)\|_X \\ &\leq (1 + \|e_n\|_\infty)\|f - g_n\|_X \leq 2K 2^{-n/p} n^{-d}\end{aligned}$$

for $n \geq 1$. If one has $d > 2$, Theorem 4 applies immediately. That $d > 1 + 1/p$ is still sufficient follows from the proof of the theorem, using the fact that a sum $\sum_{j=1}^{\infty} f_j$ is convergent in $L^p(\Omega, \Sigma, \mu)$ if $\sum_{j=1}^{\infty} \|f_j\|_p^p < \infty$ and the functions f_j are supported by pairwise disjoint, measurable sets. A similar argument applies to the additional remarks concerning the special case $X = L^q(\Omega, \Sigma, \mu)$.

Remark. k) The assumption $d > 1/p + 1$, respectively $d > 1/p + 1/q$, in the theorem is the best possible in the following sense: If we admit "$\geq$" instead of "$>$", the theorem fails. This is seen from the following example: Let $p, q \in [1, \infty)$, $d = 1/p + 1/q$ and $X = l^q(N)$. For $k \in N$ define

$$c_k = 2^{-nd} n^{-d} \quad \text{if} \quad 2^n \leq k < 2^{n+1}.$$

Let $f = \{c_k\}_{k \in N}$ and $g_n = e_n f$ where e_n is the characteristic function of $\{1, \ldots, 2^n\}$. We have

$$\|f - g_n\|_q \leq [\sum_n^{\infty} 2^k \cdot (2^{-kd} k^{-d})^q]^{1/q}$$

$$= [\sum_n^{\infty} 2^{k(1-dq)} k^{-dq}]^{1/q}$$

$$\leq n^{-d} [\sum_n^{\infty} (2^{1-dq})^k]^{1/q}$$

$$\text{(note that } 1 - dq = -q/p < 0)$$

$$\leq n^{-d} \cdot K (2^{1-dq})^{n/q} = K 2^{-n/p} n^{-d}.$$

So the assumptions of Theorem 5 are satisfied, but f cannot be written as a product since it is not in $l^{1/d}$:

$$\sum_1^{\infty} 2^k \cdot (2^{-kd} k^{-d})^{1/d} = \sum_1^{\infty} k^{-1} = \infty.$$

l) The members of $(X, \| \|_X)$ satisfying the assumption of the theorem for some fixed $d > 0$ constitute a so-called approximation space (cf. [15]). It is a quasi-Banach space with respect to a suitable quasi-norm.

m) Sequence spaces (more precisely: solid BK-spaces) are the most simple examples for the above situation.

n) It is easy to replace L^p above by a suitable Lorentz space $L(p, q)$ [2], Orlicz space [11], Lorentz–Zygmund space [1], or any other suitable rearrangement invariant solid BF-space Y on (Ω, Σ, μ) [13]. Whether such a space Y can be used or not depends on its fundamental function: $\sigma(t) := |c_M|_Y$, where M is any measurable subset of Ω with $\mu(M) = t$.

o) A careful repetition of the proof of the theorem reveals that, for example, in the case $X = L^q$ the sequence n^{-d} may even be replaced by any positive sequence $\{c_n\}_{n\geq 1} \in l^s$, $1/s = 1/p + 1/q$. If however instead of "$\{c_n\} \in l^s$" we assume "$\{c_n\} \in l^r$ for all $r > s$", the theorem fails as is seen from the example given in a).

p) Using Plancherel's theorem and the Fourier transform the special case $X = L^2(G)$ can be reformulated as follows:

COROLLARY 6. *Let G be a locally compact abelian group, and let $\hat{\mu}$ be the Haar measure on the dual group $\hat{G}$. Suppose that for $f \in L^2(G)$ there exist $\{d_n\}_{n\geq 1} \in l^1$ and a sequence $\{f_n\}_{n\geq 1}$ in $L^2(G)$ satisfying $\hat{\mu}(\operatorname{supp} \hat{f}_n) \leq 2^n$ and $\|f - f_n\|_2 \leq d_n 2^{-n/2}$. Then f belongs to the Fourier algebra $A(G) \subseteq C^0(G)$, more precisely: f is equal almost everywhere to some $f' \in A(G)$ (i.e. $\hat{f}' \in L^1(\hat{G})$).*

Remark. q) For $G = T$, the unit circle, the above result constitutes one direction of a result due to S. Stečkin (cf. [18], [15]).

A result in a similar direction is the following one:

THEOREM 7. *Let G be a compact, abelian group. Let $s \geq 2$ be fixed. If for some $f \in L^p(G)$ and a suitable constant $K > 0$ it is possible to find for each $n \in N$ a trigonometric polynomial t_n with at most 2^n nonzero (Fourier) coefficients, such that*

$$\|f - t_n\|_p \leq K 2^{-\varrho n} n^{-d} \quad \textit{for} \quad n \geq 1, \quad \varrho := 1/s' + |1/p - 1/2| \quad \textit{and some} \quad d > 2,$$

then there exist $h \in L^s(G)$ and $g \in L^p(G)$ such that $f = h * g$.

Proof. Let e_n be a trigonometric polynomial such that $e_n * t_k = t_k$ for $k = 1, 2, \ldots, n$. We may suppose that e_n has at most 2^{n+1} nonvanishing coefficients $a_j = 1$. The norm $\|e_n\|_{p\to p}$ of the convolution operator induced on $L^p(G)$ by e_n can be estimated by

$$\|e_n\|_{p\to p} \leq K_1 2^{n|1/p - 1/2|} \quad \text{for } n \geq 1.$$

The result holds for $p = 2$, since $\|e_n\|_{2\to 2} = \|\hat{e}_n\|_\infty = 1$; and for $p = 1, \infty$, since $\|e_n\|_{\infty\to\infty} = \|e_n\|_{1\to 1} \leq \|e_n\|_1 \leq \|e_n\|_2 \leq 2^{(n+1)/2}$. The general estimate follows therefrom by means of the Riesz–Thorin convexity theorem (complex interpolation, cf. [2]). Consequently

$$\begin{aligned} \|f - f * e_n\|_p &\leq \|f - t_n\|_p + \|t_n * e_n - f * e_n\|_p \\ &\leq \|f - t_n\|_p + \|t_n - f\|_p \|e_n\|_{p\to p} \\ &\leq K_2 2^{-\varrho n} n^{-d} \cdot 2^{n|1/p - 1/2|}. \end{aligned}$$

On the other hand, by the Hausdorff–Young inequality one has

$$\|e_n\|_s \leq \|\hat{e}_n\|_{s'} \leq K_3 2^{(1-1/s)n}.$$

Theorem 4 now applies to give the result.

At the end of this paper we show that the factorization theorems for Lipschitz spaces as given by various authors (cf. [3/4], [16/17], and [20/21]) may also be considered as individual factorization theorems in the above sense. For the formulation of one theorem in that direction we shall need the following notations: G will be assumed to be a totally disconnected locally compact group, with a decreasing sequence $\{G_n\}_{n\geq 1}$ of compact open subgroups, satisfying $|G_{n-1}\colon G_n|\leq M$ for $n\geq 2$, and $\bigcap_{n=1}^{\infty} G_n=\{e\}$. We write $m_n := \lambda(G_n)$ (Haar measure). X will denote a homogeneous Banach space, i.e., a Banach space, continuously embedded in $L^1_{\mathrm{loc}}(G)$ and being an essential Banach module over $L^1(G)$ with respect to convolution (or equivalently: G acts on X isometrically by translations and $y\mapsto L_y f$ is a continuous mapping from G into X for any $f\in X$ (cf. [9])). For $f\in X$ the modulus of continuity is then given by $\omega(f, n) := \sup\{\|L_y f-f\|_X \mid y\in G_n\}$. Our result now reads as follows:

THEOREM 8. *Let G and X be as above. Suppose that for some $f\in X$ there exist $d>2$ and $K>0$ such that*

$$m_n^{1/p-1}\,\omega(f, n)\leq Kn^{-d} \quad \textit{for } n\geq 1.$$

Then there exist $h\in L^p(G)$ and $g\in X$ such that $f=g*h$.

Proof. We consider X as a Banach module over the Segal algebra $(A, \|\ \|_A) := (L^1\cap L^p, \|\ \|_1+\|\ \|_p)$. If we set $e_n := m_n^{-1}c_{G_n}$, the normalized characteristic function of G_n, it is clear that $\|e_n\|_A = 1+m_n^{1/p-1} =: r_n$, and condition i) of Theorem 4 is satisfied for $k=0$. On the other hand vector-valued integration implies

$$\|e_n*f-f\|_B\leq \int_{G_n}\|L_y f-f\|_X\,|e_n(y)|\,dy\leq \omega(f, n).$$

Therefore Theorem 4 can be applied to give the result.

Remark. r) If one replaces $A=L^1\cap L^p$ by another Segal algebra $(S, \|\ \|_S)$, for which one has control of $\|e_n\|_S$ (such as the intersection of $L^1(G)$ with certain Lipschitz spaces) one may obtain several variants of the above theorem, e.g. factorizations over elements in Lipschitz spaces.

s) As in Corollary 6 factorization results can be used to derive Bernstein-type theorems for Lipschitz spaces (observing that $A(G) = L^2(G)*L^2(G)$). Such results are usually based on a characterization of the Lipschitz spaces via decompositions of the Fourier transforms of their elements. Making use of these characterizations it is also possible to derive various kinds of factorization theorems for Lipschitz spaces or Besov spaces on $\mathbf{R}^n$ (cf. [19] for a typical example).

Appendix. We include a simple proof of Cohen's theorem.

If A is a Banach algebra, let A_u be the algebra with identity u adjoined and norm $\|\lambda u+a\| = |\lambda|+\|a\|$. Let X be a left Banach-A-module with norm

$\| \|_X$ and module constant $\varkappa \geqslant 1$: $\|ax\|_X \leqslant \varkappa \|a\| \|x\|_X$. By a *left approximate identity* for X bounded by $d \geqslant 1$ we mean a net $\{e_\beta\}$ in A_u with $\|e_\beta\| \leqslant d$ and $\|e_\beta x - x\|_X \to 0$ for $x \in X$. For fixed $\gamma > 0$ we set $E_\beta = (1-\gamma)u + \gamma e_\beta$. Clearly $\{E_\beta\}$ is again a bounded left approximate identity for X. If γ is sufficiently small (such that $\frac{\gamma d}{1-\gamma} < 1$), then

$$E_\beta^{-1} = (1-\gamma)^{-1} \sum_{k=0}^{\infty} \left(\frac{\gamma e_\beta}{\gamma - 1}\right)^k$$

exists and is bounded by $(1-\gamma)^{-1} \sum_{k=0}^{\infty} \left(\frac{\gamma d}{1-\gamma}\right)^k = \text{const}$, and

$$\|E_\beta^{-1} x - x\|_X = \|E_\beta^{-1} x - E_\beta^{-1} E_\beta x\|_X \leqslant \text{const} \|x - E_\beta x\|_X \to 0,$$

so $\{E_\beta^{-1}\}$ is again a bounded left approximate identity for X. For the following we fix some small γ and the above notation E_β.

THEOREM (Cohen). *Let A be a Banach algebra, X a left Banach-A-module, and let $\{e_\beta\}$ be a net in A that is a left approximate identity for A and for X bounded by d. Then for every $x \in X$ and $\varepsilon > 0$ there are $a \in A$ and $y \in \overline{Ax}$ such that*

(i) $x = ay$;

(ii) $\|y - x\|_X < \varepsilon$.

Proof. Let $x \in X$ and $\varepsilon > 0$. We construct by induction a sequence $\{E_n\}$ such that the products $E_n \ldots E_1$ and $E_1^{-1} \ldots E_n^{-1} x$ converge to $a \in A$ and $y \in X$ respectively, thus obtaining $x = \lim_{n\to\infty} E_n \ldots E_1 \cdot E_1^{-1} \ldots E_n^{-1} x = ay$. Choose $E_1 = E_{\beta_1}$ such that $\|E_1^{-1} x - x\|_X < \varepsilon/2$. If $E_1, \ldots, E_n$ have been chosen, we have $E_n \ldots E_1 = (1-\gamma)^n u + a_n$, with $a_n \in A$, and we choose $E_{n+1} = E_{\beta_{n+1}}$ such that

$$\|E_{n+1} a_n - a_n\| < \frac{\varepsilon}{2^n} \tag{1}$$

and

$$\|E_{n+1}^{-1} x - x\|_X < \frac{\varepsilon}{\varkappa \|E_1^{-1} \ldots E_n^{-1}\| 2^{n+1}}. \tag{2}$$

Then we have

$$\|E_1^{-1} \ldots E_{n+1}^{-1} x - E_1^{-1} \ldots E_n^{-1} x\|_X \leqslant \varkappa \|E_1^{-1} \ldots E_n^{-1}\| \|E_{n+1}^{-1} x - x\|_X < \frac{\varepsilon}{2^{n+1}},$$

so $y = \lim_{n\to\infty} E_1^{-1} \ldots E_n^{-1} x$ exists and $\|y - x\|_X < \varepsilon$. Clearly $y \in \overline{Ax}$. Since

$E_n \dots E_1 = (1-\gamma)^n u + a_n$, we have

$$E_{n+1}(E_n \dots E_1) = (1-\gamma)^n E_{n+1} + E_{n+1} a_n,$$

hence

$$\|E_{n+1} \dots E_1 - E_n \dots E_1\| \leqslant (1-\gamma)^n \|E_{n+1} - u\| + \|E_{n+1} a_n - a_n\|,$$

so by (1), $a = \lim_{n \to \infty} E_n \dots E_1$ exists. We have $a \in A$ since $(1-\gamma)^n u \to 0$. Now

$$x = \lim_{n \to \infty} E_n \dots E_1 E_1^{-1} \dots E_n^{-1} x = ay.$$

Remark. a) We remind the reader that it was Cohen's fundamental idea to construct, by means of the E_i, a sequence of invertible elements $z_n \in A_u$ such that $z_n \mapsto a \in A$ (note that a is not invertible) and $z_n^{-1} x$ converges, too. It was Koosis (cf. [10]) who considerably simplified the proof by noting that z_n could be chosen in the form $E_1 \dots E_n$. We use $E_n \dots E_1$ instead, in order not to need a two-sided approximate indentity, and we do not return to the e_i in the proof (except for Remark d) below where it is necessary) since working with the E_i makes things more obvious.

b) The approximations taken in the proof are canonical because $\{E_\beta\}$ and $\{E_\beta^{-1}\}$ are left approximate identities. It is clear that a_n rather than $E_n \dots E_1$ has to occur in (1), since $\{E_\beta\}$ is an approximate identity for A but not for A_u. So, once the starting-point $x = E_n \dots E_1 E_1^{-1} \dots E_n^{-1} x$ has been chosen, the proof is automatic, if one just uses the obvious facts that $\{E_\beta\}$ and $\{E_\beta^{-1}\}$ are bounded left approximate identities.

c) It is clear that the proof above also works in the operator setting of [6].

d) The assertion

(iii) $$\|a\| \leqslant d$$

is usually included in the above theorem (and has been proved by Cohen, too, without being explicitly stated). Assumed (i) and (ii), it is clear that (iii) is implied by

(iii′) $$\|a\| \leqslant d + \varepsilon$$

since $x = ay$ may be written as

$$x = \frac{d}{d+\varepsilon} a \cdot \frac{d+\varepsilon}{d} y.$$

We obtain (iii′) together with (i) and (ii) if, in the proof of the theorem, we replace the last three sentences by the following: Since $E_n \dots E_1 = (1-\gamma)^n u + a_n$, we have

$$E_{n+1} \dots E_1 = (1-\gamma)^n E_{n+1} + E_{n+1} a_n$$

whose A-component a_{n+1} is $(1-\gamma)^n \gamma e_{\beta_{n+1}} + E_{n+1} a_n$, so by (1) we have

$$\|a_{n+1} \dots a_n\| < (1-\gamma)^n \gamma d + \frac{\varepsilon}{2^n}.$$

Hence $a = \lim a_n (= \lim E_n \dots E_1)$ exists and

$$\|a\| < \|a_1\| + \sum_{n=1}^{\infty} (1-\gamma)^n \gamma d + \varepsilon \leqslant d + \varepsilon$$

because $\|a_1\| = \|\gamma e_{\beta_1}\| \leqslant \gamma d$. We have

$$x = \lim E_n \dots E_1 \cdot E_1^{-1} \dots E_n^{-1} x = ay.$$

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