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ON A COREGULAR DIVISION OF A DIFFERENTIAL SPACE BY AN EQUIVALENCE RELATION

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Introduction. Given a set M and an equivalence relation R on M, we denote by $p_R(x)$ the equivalence class of R containing x and by M/R the set of all cosets $p_R(x)$, $x \in M$. A well-known Godement theorem gives a necessary and sufficient condition for M/R to have the structure of a differentiable manifold such that the *natural mapping* $p_R\colon M \to M/R$ be a submersion. In the present paper we give a similar condition in the category of differential spaces. In the category of differentiable manifolds certain conditions should be satisfied in order that the natural mapping p_R be a smooth one. In the category of differential spaces assuring special conditions for smoothness of the natural mapping is not a problem. It is easy to prove (cf. [4] and [5]) that for a given differential space $(M, \mathscr{C})$ and for any mapping f of the set M into the set N there is the smallest differential structure $\mathscr{D}$ on N such that f is a smooth mapping of $(M, \mathscr{C})$ into $(N, \mathscr{D})$. The structure $\mathscr{D}$ is not determined by the requirement of smoothness of the mapping f of $(M, \mathscr{C})$ into $(N, \mathscr{D})$. Similarly as in the theory of differentiable manifolds such a uniqueness assures the requirement for the p_R to be a submersion.

1. Preliminaries. For any function f and for any set A contained in the domain of f we denote by $f|A$ and $f[A]$ the restriction of f to A and the image of A given by f, respectively. For any set B we denote by $f^{-1}[B]$ the inverse image of B given by f, i.e., the set of all x of the domain of f such that $f(x) \in B$. Let M be any set and $\mathscr{C}$ an arbitrary set of real functions defined on M. By $\tau_{\mathscr{C}}$ we denote the weakest topology on M such that all functions belonging to $\mathscr{C}$ are continuous. So, the set B is *open* in the topology $\tau_{\mathscr{C}}$ if and only if for any point x of B there exist functions $a_1, \ldots, a_m$ of $\mathscr{C}$ and real numbers $a_1, b_1, \ldots, a_m, b_m$ such that

$$(1) \qquad x \in \bigcap_{i=1}^{m} a_i^{-1}[(a_i; b_i)] \subset B.$$

For every topological space X and for every set A of points of X we denote by $X|A$ the topological space induced on the set A by the topological space X. For any set $\mathscr{C}$ of real functions defined on M and for any set A contained in M we denote by $\mathscr{C}|A$ the set of all functions of the form $\alpha|A$, where $\alpha\in\mathscr{C}$. By $\mathscr{C}_A$ we denote the set of all functions $\beta\colon A\to\boldsymbol{R}$ such that for any point x of A there exist a neighbourhood U of x open in $\tau_{\mathscr{C}}$ and a function α of $\mathscr{C}$ such that $\beta|A\cap U = \alpha|A\cap U$. It is easy to verify, making use of condition (1), that for any set $A\subset M$ we have $\tau_{\mathscr{C}_A} = \tau_{\mathscr{C}|A} = \tau_{\mathscr{C}}|A$. In particular, we obtain $\tau_{\mathscr{C}_M} = \tau_{\mathscr{C}}$.

We denote by $\mathrm{sc}\mathscr{C}$ the set of all real functions of the form

$$\varphi\big(\alpha_1(\cdot),\ldots,\alpha_m(\cdot)\big),$$

where φ is a real function of the class $C^\infty(\boldsymbol{R}^m)$, $\alpha_1,\ldots,\alpha_m\in C$ and m is an arbitrary positive integer. The ordered pair $(M,\mathscr{C})$ such that $\mathscr{C}_M = \mathscr{C} = \mathrm{sc}\mathscr{C}$ is said to be a *differential space* (cf. [1] and [4]) or, shortly, *space.* The set $\mathscr{C}$ is called a *differential structure* for this space. This definition is equivalent to that given by Sikorski in [3]. For any set $\mathscr{C}$ of real functions defined on M, the set $(\mathrm{sc}\mathscr{C})_M$ is the smallest among all sets $\mathscr{C}'$ such that $\mathscr{C}\subset\mathscr{C}'$ and $(M,\mathscr{C}')$ is a differential space. $\big(M,(\mathrm{sc}\mathscr{C})_M\big)$ is called the *differential space generated by* $\mathscr{C}$ (cf. [1] and [5]).

For any function f such that the set of all values is contained in M and for any set $\mathscr{C}$ of functions defined on M, we denote by $\mathscr{C}f$ the set of all functions of the form $\alpha\circ f$, where $\alpha\in\mathscr{C}$. Let $(M_1,\mathscr{C}_1)$ and $(M_2,\mathscr{C}_2)$ be any spaces. By the *Cartesian product* $(M_1,\mathscr{C}_1)\times(M_2,\mathscr{C}_2)$ we mean the space generated by the set

$$\mathscr{C}_1\mathrm{pr}_1|M_1\times M_2\cup\mathscr{C}_2\mathrm{pr}_2|M_1\times M_2.$$

We denote the differential structure of this space by $\mathscr{C}_1\times\mathscr{C}_2$. Then we have

$$\mathscr{C}_1\times\mathscr{C}_2 = \big(\mathrm{sc}(\mathscr{C}_1\mathrm{pr}_1|M_1\times M_2\cup\mathscr{C}_2\mathrm{pr}_2|M_1\times M_2)\big)_{M_1\times M_2}\tag{2}$$

and

$$(M_1,\mathscr{C}_1)\times(M_2,\mathscr{C}_2) = (M_1\times M_2,\mathscr{C}_1\times\mathscr{C}_2).$$

Let $(M,\mathscr{C})$ and $(N,\mathscr{D})$ be differential spaces. We say that a function f maps *smoothly* $(M,\mathscr{C})$ into $(N,\mathscr{D})$ which we note in the form

$$f\colon(M,\mathscr{C})\to(N,\mathscr{D}),\tag{3}$$

iff f maps the set M into N and, for any function β of $\mathscr{D}$, the function $\beta\circ f$ belongs to $\mathscr{C}$. Mapping (3) is called a *diffeomorphism* iff f maps the set M one-to-one and onto N and $f^{-1}\colon(N,\mathscr{D})\to(M,\mathscr{C})$. It is easy to verify that if a function f maps smoothly $(M,\mathscr{C})$ into $(N,\mathscr{D})$, then f is a continuous mapping of the topological space $\tau_{\mathscr{C}}$ into the topological space $\tau_{\mathscr{D}}$ and, consequently, if mapping (3) is a diffeomorphism, then f s a homeomorphism of $\tau_{\mathscr{C}}$ onto $\tau_{\mathscr{D}}$. Next, it is easy to prove that

$$f\colon (M, \mathscr{C}) \to (M_1, \mathscr{C}_1) \times (M_2, \mathscr{C}_2)$$

if and only if

$$\mathrm{pr}_i \circ f\colon (M, \mathscr{C}) \to (M_i, \mathscr{C}_i), \quad i = 1, 2.$$

Smooth mapping (3) will be called *regular at a point* x iff there exist a neighbourhood U of x open in $\tau_{\mathscr{C}}$, a space (M_0, C_0), a point $m_0 \in M_0$, a set V open in $\tau_{\mathscr{D}}$ such that $f[U] \subset V$, and a diffeomorphism

$$\varphi\colon (U, \mathscr{C}_U) \times (M_0, C_0) \to (V, \mathscr{D}_V) \tag{4}$$

such that $\varphi \circ i = f | U$, where $i(u) = (u, m_0)$ for $u \in U$. Mapping (3) which is regular at every point x of M will be called *regular* (or else — an *immersion*). We say that a *space* $(M', \mathscr{C}')$ *is lying regularly* in $(M, \mathscr{C})$ iff $M' \subset M$ and the inclusion

$$\mathrm{id}_{M'}\colon (M', \mathscr{C}') \to (M, \mathscr{C}) \tag{5}$$

is a regular mapping. We say that a *set* $A \subset M$ *is lying regularly* in a space $(M, \mathscr{C})$ iff the space $(A, \mathscr{C}_A)$ is lying regularly in $(M, \mathscr{C})$.

Smooth mapping (3) will be called *coregular at a point* x iff there exist a neighbourhood U of x open in $\tau_{\mathscr{C}}$, a set V open in $\tau_{\mathscr{D}}$ such that $f[U] \subset V$, a differential space (N_0, D_0), and a diffeomorphism

$$\varphi\colon (U, \mathscr{C}_U) \to (V, \mathscr{D}_V) \times (N_0, \mathscr{D}_0) \tag{6}$$

such that $\mathrm{pr}_1 \circ \varphi = f | U$. Mapping (3) which is coregular at every point x of M will be called *coregular* (or else — a *submersion*). In [5] it is proved that

1.1. *If there exists a coregular mapping* (3) *such that the mapping*

$$f\colon (M, \mathscr{C}) \to (N, \mathscr{D}') \tag{7}$$

is also coregular and f maps M onto N, then $(N, \mathscr{D})$ is equal to $(N, \mathscr{D}')$.

Consider now an arbitrary mapping

$$f\colon M \to N. \tag{8}$$

For any real function β defined on the set N we put $f^*(\beta) = \beta \circ f$.

Let $\mathscr{C}$ and $\mathscr{D}$ be any sets of real functions defined on M and N, respectively. In [5] it is proved that

1.2. *If $(M, \mathscr{C})$ and $(N, \mathscr{D})$ are differential spaces, then $(f^{*-1}[\mathscr{C}])_N$ is the greatest set among sets $\mathscr{D}'$ such that $(N, \mathscr{D}')$ is a differential space and mapping* (7) *is smooth. Similarly, $(f^*[\mathscr{D}])_M$ is the smallest set among sets $\mathscr{C}'$ such that $(M, \mathscr{C}')$ is a differential space and the mapping $f\colon (M, \mathscr{C}') \to (N, \mathscr{D})$ is smooth.*

The space $(M, (f^*[\mathscr{D}])_M)$ will be called *induced* on M from $(N, \mathscr{D})$ by mapping (8). Similarly, the space $(N, (f^{*-1}[\mathscr{C}])_N)$ will be called *coinduced* on N from $(M, \mathscr{C})$ by mapping (8). For any set A contained in M, the differential space induced from $(M, \mathscr{C})$ by the mapping $\mathrm{id}_A\colon A \to M$ coincides with $(A, \mathscr{C}_A)$ and is called a *subspace of* $(M, \mathscr{C})$ *induced on* A. It is easy to check that

1.3. *A set W is lying regularly in $(U, \mathscr{C}_U)$, where U is an open set in $\tau_{\mathscr{C}}$, if and only if $W \subset U$ and W is lying regularly in $(M, \mathscr{C})$.*

For any subset B of N and for any mapping (3), the subspace of $(M, \mathscr{C})$ induced on $f^{-1}[B]$ will be called the *inverse image* of the subspace $(B, \mathscr{D}_B)$ of $(N, \mathscr{D})$ given by mapping (3). Now, we prove the following lemma:

1.4. *The inverse image of a lying regularly subspace of $(N, \mathscr{D})$ given by a coregular mapping (3) is lying regularly in $(M, \mathscr{C})$.*

Proof. Let us consider any point $x_0 \in f^{-1}[B]$, where B is lying regularly in $(N, \mathscr{D})$, and set $y_0 = f(x_0)$. Then there exist a neighbourhood U of x_0 open in $\tau_{\mathscr{C}}$, a neighbourhood V of point y_0 open in $\tau_{\mathscr{D}}$, a differential space $(N_0, \mathscr{D}_0)$, and a diffeomorphism (6) such that $\mathrm{pr}_1 \circ \varphi = f \mid U$. From the hypothesis that the space $(B, \mathscr{D}_B)$ is lying regularly in $(N, \mathscr{D})$ it follows that there exist neighbourhoods V_1 and W of y_0 open in topological spaces $\tau_{\mathscr{D}}$ and $\tau_{\mathscr{D}_B}$, respectively, a space $(N_1, \mathscr{D}_1)$, a point n_1 of this space, and a diffeomorphism

$$\psi\colon (V_1, \mathscr{D}_{V_1}) \to (W, \mathscr{D}_W) \times (N_1, \mathscr{D}_1)$$

such that $W \subset V_1$ and $\psi(w) = (w, n_1)$ for $w \in W$. The set $\psi[V \cap V_1]$ is open in the topological space $\tau_{\mathscr{D}_W} \times \tau_{\mathscr{D}_1}$. Then there exist a neighbourhood W_1 of the point y_0 open in $\tau_{\mathscr{D}_W} = \tau_{\mathscr{D}} | W$ and a neighbourhood N_1' of the point n_1 open in $\tau_{\mathscr{D}_1}$ such that the set $W_1 \times N_1'$ is contained in $\psi[V \cap V_1]$. Setting $V_1' = \psi^{-1}[W_1 \times N_1']$, we get the diffeomorphism

$$\psi \mid V_1'\colon (V_1', \mathscr{D}_{V_1'}) \to (W_1, \mathscr{D}_{W_1}) \times (N_1', (\mathscr{D}_1)_{N_1'}), \tag{9}$$

where V_1' is open in $\tau_{\mathscr{D}} \mid V_1$. Next, let us put $U' = \varphi^{-1}[V_1' \times N_0]$. The set U' is a neighbourhood of x_0 open in $\tau_{\mathscr{C}}$ and we have the diffeomorphism

$$\varphi \mid U'\colon (U', \mathscr{C}_{U'}) \to (V_1', \mathscr{D}_{V_1'}) \times (N_0, \mathscr{D}_0) \tag{10}$$

fulfilling the equality $f \mid U' = \mathrm{pr}_1 \circ \varphi \mid U'$. From (9) we obtain the diffeomorphism $\psi \mid V_1' \times \mathrm{id}_{N_0}$ of the space $(V_1', \mathscr{D}_{V_1'}) \times (N_0, \mathscr{D}_0)$ onto the space

$$\big((W_1, \mathscr{D}_{W_1}) \times (N_1', (\mathscr{D}_1)_{N_1'})\big) \times (N_0, \mathscr{D}_0). \tag{11}$$

Denote by $\varkappa$ the natural mapping of the set $(W_1 \times N_1') \times N_0$ onto the set $(W_1 \times N_0) \times N_1'$. Then $\varkappa$ is a diffeomorphism of the space (11) onto the space

$$\big((W_1, \mathscr{D}_{W_1}) \times (N_0, D_0)\big) \times (N_1', (\mathscr{D}_1)_{N_1'}). \tag{12}$$

Next, setting

$$Q = \varphi^{-1}[W_1 \times N_0] \tag{13}$$

and making use of (10), we get the diffeomorphism $(\varphi|Q\times \mathrm{id}_{N_1'})^{-1}$ of the space (12) onto the space $(Q, \mathscr{C}_Q)\times (N_1', (\mathscr{D}_1)_{N_1'})$. Setting

$$\eta = (\varphi|U')^{-1}\circ(\psi|V_1'\times \mathrm{id}_{N_0})^{-1}\circ\varkappa^{-1}\circ(\varphi|Q\times \mathrm{id}_{N_1'}),$$

we obtain the diffeomorphism

$$\eta\colon (Q, \mathscr{C}_Q)\times (N_1', (\mathscr{D}_1)_{N_1'}) \to (Q, \mathscr{C}_Q)$$

satisfying, as it is easy to verify, the condition $\eta(u, n_1) = u$ for $u\in Q$. From definition (13) of the set Q it follows that $Q = U\cap f^{-1}[W_1]$. Since W_1 is open in $\tau_{\mathscr{D}_W}$, Q is open in $\tau_{\mathscr{C}_A}$, where $A = f^{-1}[B]$. Thus the mapping $\mathrm{id}_A\colon (A, \mathscr{C}_A)\to (M, \mathscr{C})$ is regular at x_0. Thus the space $(A, \mathscr{C}_A)$ is lying regularly in $(M, \mathscr{C})$.

1.5. *If mapping* (3) *is coregular and R is the set of all pairs $(x, y)\in M\times M$ such that $f(x) = f(y)$, then the mapping*

$$\mathrm{pr}_1|R\colon (R, (\mathscr{C}\times\mathscr{C})_R) \to (M, \mathscr{C}) \tag{14}$$

is coregular.

Proof. Let $(x_0, y_0)\in R$. From the coregularity of mapping (3) it follows that there exist a neighbourhood V of the point $z_0 = f(x_0)$ open in $\tau_{\mathscr{D}}$, neighbourhoods U_1 and U_2 of the points x_0 and y_0, respectively, open in $\tau_{\mathscr{C}}$, spaces $(N_1, \mathscr{D}_1)$ and $(N_2, \mathscr{D}_2)$, and diffeomorphisms

$$\varphi_i\colon (U_i, \mathscr{C}_{U_i}) \to (V, \mathscr{D}_V)\times (N_i, \mathscr{D}_i) \tag{15}$$

such that $f|U_i = \mathrm{pr}_1\circ\varphi_i$, $i = 1, 2$. Set

$$\varphi_i(x) = (f(x), h_i(x)) \quad \text{for } x\in U_i,\ i = 1, 2. \tag{16}$$

Then we have $h_i\colon (U_i, \mathscr{C}_{U_i}) \to (N_i, \mathscr{D}_i)$, $i = 1, 2$. Next, we set

$$\theta(x, y) = (x, h_2(y)) \quad \text{for } (x, y)\in (U_1\times U_2)\cap R. \tag{17}$$

From (16) and (17) it follows that the function θ is one-to-one and

$$\theta^{-1}(x, t) = (x, \varphi_2^{-1}(f(x), t)) \quad \text{for } (x, t)\in U_1\times N_2.$$

Then the function θ is a diffeomorphism of the differential space

$$((U_1\times U_2)\cap R, (\mathscr{C}\times\mathscr{C})_{(U_1\times U_2)\cap R})$$

onto the space $(U_1, \mathscr{C}_{U_1})\times (N_2, \mathscr{D}_2)$. Equality (17) yields

$$\mathrm{pr}_1|(U_1\times U_2)\cap R = \mathrm{pr}_1|(U_1\times N_2)\circ\theta.$$

Mapping (14) is then coregular at the point (x_0, y_0). This completes the proof.

The following lemma will be useful in the proof of next lemma concerning coregularity of a mapping f of a space $(M, \mathscr{C})$ into the space coinduced from $(M, \mathscr{C})$ by (3):

1.6. *If function f maps set M into set N, $(M, \mathscr{C})$ is a differential space, sets U and $f[U]$ are open in topological spaces $\tau_{\mathscr{C}}$ and $\tau_{f^{*-1}[\mathscr{C}]}$, respectively, and $f^{-1}[f[U]] = U$, then the equality*

$$(18)\qquad \big(f[U], (f^{*-1}[\mathscr{C}])_{f[U]}\big) = \big(f[U], ((f\,|\,U)^{*-1}[\mathscr{C}_U])_{f[U]}\big)$$

holds.

Proof. Let us take any function $\beta \in f^{*-1}[\mathscr{C}]\,|f[U]$, i.e., $\beta = \gamma\,|f[U]$, where $\gamma: N \to \boldsymbol{R}$ and $\gamma \circ f \in \mathscr{C}$. Hence, we obtain

$$\beta \circ f\,|\,U = \gamma\,|f[U] \circ f\,|\,U = \gamma \circ f\,|\,U \in \mathscr{C}\,|\,U \subset \mathscr{C}_U.$$

Then $\beta \in (f\,|\,U)^{*-1}[\mathscr{C}_U]$. Thus we have proved the inclusion

$$f^{*-1}[\mathscr{C}]\,|f[U] \subset (f\,|\,U)^{*-1}[\mathscr{C}_U].$$

Making use of this inclusion, we get

$$(f^{*-1}[\mathscr{C}])_{f[U]} = (f^{*-1}[\mathscr{C}]\,|f[U])_{f[U]} \subset \big((f\,|\,U)^{*-1}[\mathscr{C}_U]\big)_{f[U]}.$$

Now, let us consider any function $\beta \in (f\,|\,U)^{*-1}[\mathscr{C}_U]$, i.e., $\beta: f[U] \to \boldsymbol{R}$ and $\beta \circ f\,|\,U \in \mathscr{C}_U$. Let $y_0 \in f[U]$. Since $f[U]$ is open in the topological space $\tau_{f^{*-1}[\mathscr{C}]}$, there exist functions $\beta_i \in f^{*-1}[\mathscr{C}]$ and real numbers a_i and b_i, $i = 1, \ldots, m$, such that

$$y_0 \in \bigcap_{i=1}^{m} \beta_i^{-1}[(a_i; b_i)] \subset f[U].$$

Then there exist a_i', a_i'', b_i'', b_i' such that

$$(19)\qquad a_i < a_i' < a_i'' < \beta_i(y_0) < b_i'' < b_i' < b_i, \qquad i = 1, \ldots, m.$$

There are real functions η_i, infinitely differentiable on $\boldsymbol{R}$, such that $\eta_i(t) = 1$ for $t \in (a_i''; b_i'')$, and $\eta_i(t) = 0$ for $t \in \boldsymbol{R} - (a_i'; b_i')$. Setting

$$\eta(y) = \prod_{i=1}^{m} \eta_i\big(\beta_i(y)\big) \qquad \text{for } y \in N,$$

we obtain the function $\eta: N \to \boldsymbol{R}$ such that

$$\eta \circ f = \prod_i \eta_i \circ \beta_i \circ f \in \mathscr{C},$$

because $\eta_i \circ f \in \mathscr{C}$, and η fulfis conditions $\eta(y) = 1$ for $y \in V$, where

$$V = \bigcap_i \beta_i^{-1}[(a_i''; b_i'')] \qquad \text{and} \qquad \eta(y) = 0$$

for

$$y \in N - \bigcap_i \beta_i^{-1}[(a_i'; b_i')].$$

Put

$$\lambda(y) = \begin{cases} \eta(y)\beta(y) & \text{for } y \in f[U], \\ 0 & \text{for } y \in N - f[U]. \end{cases}$$

Then we have the equality

$$\lambda | V = \beta | V. \tag{20}$$

Let us set

$$W = N - \bigcap_{i=1}^{m} \beta_i^{-1}[\langle a_i'; b_i' \rangle].$$

Since functions β_i map the topological space $\tau_{f_*^{-1}[\mathscr{C}]}$ into $\boldsymbol{R}$ continuously, the sets $\beta_i^{-1}[\langle a_i'; b_i' \rangle]$ are closed in it. Therefore, the defined above set W is open in $\tau_{f_*^{-1}[\mathscr{C}]}$. Evidently, we have $W \cap f[U] = N$. Hence

$$f^{-1}[W] \cup U = f^{-1}[W] \cup f^{-1}\big[f[U]\big] = f^{-1}[N] = M.$$

In other words, sets $f^{-1}[W]$ and U make a covering of M open in $\tau_{\mathscr{C}}$. Notice that $\lambda \circ f | U = \eta \circ f | U \cdot \beta \circ f | U \in \mathscr{C}_U$, because $\eta \circ f | U \in \mathscr{C}_U$. Definition of λ yields $\lambda(y) = 0$ for $y \in W$. Hence $\lambda\big(f(x)\big) = 0$ for $x \in f^{-1}[W]$. Thus $\lambda \circ f | f^{-1}[W] \in \mathscr{C}_{f^{-1}[W]}$. Therefore, we have $\lambda \circ f \in \mathscr{C}$. In other words, $\lambda \in f^{*-1}[C]$. From inequalities (19) it follows that $y_0 \in V$, where V is open in $\tau_{f_*^{-1}[\mathscr{C}]}$ and contained in $f[U]$. According to (20), the function β belongs to the set $(f^{*-1}[\mathscr{C}])_{f[U]}$. Thus we got the inclusion

$$(f|U)^{*-1}[\mathscr{C}_U] \subset (f^{*-1}[\mathscr{C}])_{f[U]}.$$

Then (18) is fulfilled. As an easy consequence of Lemma 1.6, we get

1.7. *If $f\colon M \to N$, $(M, \mathscr{C})$ is a differential space, $\mathfrak{D}$ is a covering of M open in $\tau_{\mathscr{C}}$ such that all mappings*

$$f|U\colon (U, \mathscr{C}_U) \to \big(f[U], \big((f|U)^{*-1}[\mathscr{C}_U]\big)_{f[U]}\big) \tag{21}$$

are coregular, $f[U]$ is open in $\tau_{f_*^{-1}[\mathscr{C}]}$, and $f^{-1}\big[f[U]\big] = U$ when $U \in \mathfrak{D}$, then

$$f\colon (M, \mathscr{C}) \to \big(N, (f^{*-1}[\mathscr{C}])_N\big) \tag{22}$$

is coregular.

2. Coregularity of natural mapping of an equivalence relation. Let (M, C) be an arbitrary differential space and R be any equivalence relation. By $(M, \mathscr{C})/R$ we denote the differential space $\big(M/R, (p_R^{*-1}[\mathscr{C}])_{M/R}\big)$, where p_R is the natural mapping defined by R. In this section we prove the following theorem:

2.1. *The mapping*

$$p_R\colon (M, \mathscr{C}) \to (M, \mathscr{C})/R \tag{23}$$

is coregular if and only if the following conditions are fulfilled:

(a) *The mapping* $\mathrm{pr}_1|R\colon \big(R, (\mathscr{C}\times\mathscr{C})_R\big) \to (M, \mathscr{C})$ *is coregular.*

(b) *For any point* $x_0 \in M$ *there exist a neighbourhood* U *of* x_0 *and a coregular mapping*

$$s\colon (U, \mathscr{C}_U) \to (W, \mathscr{C}_W) \tag{24}$$

such that

$$W \cap p_R(u) = \{s(u)\} \quad \textit{for } u \in U. \tag{25}$$

Proof. Necessity. Condition (a) is a direct consequence of Lemma 1.5. Let us take any $x_0 \in M$. Then there exist a neighbourhood U of x_0 open in $\tau_{\mathscr{C}}$, a neighbourhood V of the point $z_0 = p_R(x_0)$ open in the topological space $\tau_{\mathscr{C}}/R$, a differential space $(N_0, \mathscr{D}_0)$, and a diffeomorphism φ of the space $(U, \mathscr{C}_U)$ onto the space $(V, \mathscr{D}_V)\times(N_0, \mathscr{D}_0)$ such that $p_R|U = \mathrm{pr}_1\circ\varphi$. Let us set $W = \varphi^{-1}[V\times\{n_0\}]$, where $\varphi(x_0) = (z_0, n_0)$, $n_0 \in N$, and $s(u) = \varphi^{-1}\big(p_R(u), n_0\big)$ for $u \in U$. It is easy to see that the function s fulfils condition (25). Setting

$$\psi(u) = \big(s(u), \mathrm{pr}_2(\varphi(u))\big) \quad \text{for } u \in U,$$

we obtain, as it is easy to verify, the diffeomorphism

$$\psi\colon (U, \mathscr{C}_U) \to (W, \mathscr{C}_W)\times(N_0, \mathscr{D}_0)$$

such that $\mathrm{pr}_1\circ\psi = s$. Then function s gives a coregular mapping of the space $(U, \mathscr{C}_U)$ onto $(W, \mathscr{C}_W)$. Condition (b) is thus satisfied.

Before the proof of sufficiency of conditions (a) and (b) we prove two lemmas.

2.2. *If an equivalence relation* R *satisfies conditions* (a) *and* (b) *and there exists a set* U *open in* $\tau_{\mathscr{C}}$ *and such that*

$$p_R^{-1}\big[p_R[U]\big] = M, \tag{26}$$

and if the natural mapping

$$p_{R_U}\colon (U, \mathscr{C}_U) \to (U, \mathscr{C}_U)/R_U \tag{27}$$

of the relation $R_U = R\cap(U\times U)$ *is coregular, then mapping* (23) *is coregular.*

Proof. If $p_{R_U}(u) = p_{R_U}(u')$, $u, u' \in U$, then $p_R(u) = p_R(u')$. Then there exists exactly one mapping

$$h\colon U/R_U \to M/R \tag{28}$$

such that $h \circ p_{R_U} = p_R | U$. Making use of (26), it is easy to see that mapping (28) is one-to-one and onto. Let $(M/R, \mathscr{D})$ be the differential space coinduced in the set M/R from the space $(U, \mathscr{C}_U)/R_U$ by mapping (28). Hence we obtain the diffeomorphism

$$(29) \qquad h: (U, \mathscr{C}_U)/R_U \to (M/R, \mathscr{D}).$$

Let us set $k = h^{-1} \circ p_R$. It is easy to check that

$$(30) \qquad k \circ \mathrm{pr}_2 | (U \times M) \cap R = p_{R_U} \circ \mathrm{pr}_1 | (U \times M) \cap R.$$

From (a) and symmetry of R it follows that the mapping

$$\mathrm{pr}_2 | R: \big(R, (\mathscr{C} \times \mathscr{C})_R\big) \to (M, \mathscr{C})$$

is coregular. Then also

$$\mathrm{pr}_2 | (U \times M) \cap R: \big((U \times M) \cap R, (\mathscr{C} \times \mathscr{C})_{(U \times M) \cap R}\big) \to (M, \mathscr{C})$$

is coregular. Equality (30) and coregularity of (27) yields (cf. [5]) coregularity of $k: (M, \mathscr{C}) \to (U, \mathscr{C}_U)/R_U$.

The mapping $p_R: (M, \mathscr{C}) \to (M/R, \mathscr{D})$ is then coregular. Hence we have the identity of spaces $(M, \mathscr{C})/R$ and $(M/R, \mathscr{D})$. Thus (23) is coregular.

2.3. *If a set U open in $\tau_{\mathscr{C}}$ is such that there exist a set $W \subset U$ and a coregular mapping* (24) *satisfying* (25), *then the natural mapping* (27) *is coregular.*

Proof. From (25) it follows that if $s(u) = s(u')$, $u, u' \in U$, then $p_R(u) = p_R(u')$, and so $p_{R_U}(u) = p_{R_U}(u')$. Hence we obtain the mapping

$$(31) \qquad l: W \to U/R_U$$

such that

$$(32) \qquad l \circ s = p_{R_U}.$$

If $w \in W$, then $w \in W \cap p_R(w)$. So, by (25), $s(w) = w$. Therefore, if $l(w) = l(w')$, $w, w' \in W$, then, by (32), $p_R(w) = p_R(w')$. Hence and from (25) we get $s(w) = s(w')$. Therefore, $w = w'$. Let $(U/R_U, \mathscr{D})$ be the space coinduced from $(W, \mathscr{C}_W)$ by mapping (31). According to (32), we get the coregular mapping

$$p_{R_U}: (U, \mathscr{C}_U) \to (U/R_U, \mathscr{D}).$$

Then $(U/R_U, \mathscr{D}) = (U, \mathscr{C}_U/R_U)$. Hence (27) is coregular.

Now we shall finish the proof of Theorem 2.1.

Sufficiency. Let us suppose (a) and (b). From lemma 2.3 it follows that for every point x of M there exists a neighbourhood U'_x of x open in $\tau_{\mathscr{C}}$ such that

(33) $$p_{R_{U'_x}}\colon (U'_x, \mathscr{C}_{U'_x}) \to (U'_x, \mathscr{C}_{U'_x})/R_{U'_x}$$

is coregular. Let us set

(34) $$U_x = p_R^{-1}[p_R[U'_x]] \quad \text{for } x \in M.$$

By a simple verification we infer that

(35) $$U_x = \mathrm{pr}_1[R \cap (U'_x \times M)].$$

Condition (a) yields openness of U_x in $\tau_{\mathscr{C}}$ and this, together with equality (34), leads to

$$U_x = p_{R_{U_x}}^{-1}[p_{R_{U_x}}[U'_x]]$$

and to a coregular mapping

$$\mathrm{pr}_1 | R_{U_x}\colon \big(R_{U_x}, (\mathscr{C}_{U_x} \times \mathscr{C}_{U_x})_{R_{U_x}}\big) \to (U_x, \mathscr{C}_{U_x}),$$

because for every set U contained in M we have

$$(\mathscr{C} \times \mathscr{C})_{U \times U} = (\mathscr{C}_U \times \mathscr{C}_U)_{U \times U}.$$

Equality (34) yields $U_x = p_R^{-1}[p_R[U_x]]$. Hence $p_R | U_x = p_{R_{U_x}}$ and from lemma 2.2 we get the coregular mapping

$$p_R | U_x\colon (U_x, \mathscr{C}_{U_x}) \to (U_x, \mathscr{C}_{U_x})/R_{U_x}.$$

Since lemma 1.7 yields coregularity of (23), the proof of Theorem 2.1 is completed.

Let $\Delta(N) = \{(x, x); x \in N\}$, where N is a set. Note that for every equivalence relation R on a set M we have the equality

$$R = (p_R \times p_R)^{-1}[\Delta(M/R)].$$

As a corollary of 1.7, we get

2.4. *If a set $\Delta(M/R)$ is lying regularly in a differential space $(M, \mathscr{C})/R \times \times (M, \mathscr{C})/R$, then the set R is lying regularly in $(M, \mathscr{C}) \times (M, \mathscr{C})$.*

3. Comparison of regularity and coregularity of mappings of differential spaces and of manifolds. Let $(M, \mathscr{C})$ be a differentiable manifold which can be meant as a differential space locally diffeomorphic to a Euclidean space (cf. [4]). It is easy to see that if (3) is a regular (respectively, coregular) mapping, then (3) is weak regular (respectively, weak coregular), i.e., for any point x of M there exist a neighbourhood U of x open in $\tau_{\mathscr{C}}$, a neighbourhood V of $f(x)$ open in $\tau_{\mathscr{D}}$, and a smooth mapping $\sigma\colon (V, \mathscr{D}_V) \to (U, \mathscr{C}_U)$ such that $\sigma \circ f = \mathrm{id}_U$ (respectively, $f \circ \sigma = \mathrm{id}_V$). If

$(M, \mathscr{C})$ and $(N, \mathscr{D})$ are manifolds, then weak regularity (respectively, weak coregularity) is necessary and sufficient in order that (3) be an immersion (respectively, a submersion) (cf. [2]). Then, in this case, the concept of a regular mapping coincides with the concept of an immersion of differentiable manifolds. Similarly, the concept of a coregular mapping coincides with that of a submersion of differentiable manifolds. In particular, the concept of a submanifold $(M, \mathscr{C})$ coincides with that of a manifold lying regularly in $(M, \mathscr{C})$.

In [2] there is a proof of Godement's theorem on the division of a differentiable manifold by an equivalence relation R. This theorem says that

There is a manifold Y such that the mapping $p_R\colon X \to Y$ is coregular if and only if (i) R determines a submanifold of $X \times X$ and (ii) pr_1 maps R, treated as a submanifold of $X \times X$, coregularly into Y.

The proof in [2] holds for manifolds which are analytic with respect to any normed field. But the character of this proof is such that it may be verbally repeated for real manifolds of the class $C^\infty(\boldsymbol{R})$. It is evident that if Y is a differentiable manifold, then the diagonal $\Delta(Y')$ of the set Y' of all points of Y is lying regularly in $Y \times Y$. From 2.4 it follows that (i) holds, and condition (ii) coincides with (a). Condition (b) is a consequence of (i) and (ii) (cf. the proof of Godement's theorem in [2]).

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