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PARTITIONING SPACES INTO HOMEOMORPHIC RIGID PARTS

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Let X be a topologically complete (separable metric) space which is dense in itself and which in addition admits a fixed-point free involution. Then X can be decomposed into two sets A and B such that A is rigid, A is homeomorphic to B, and A contains no uncountable compact subsets. If X is a Peano continuum in which no countable set separates a nonempty connected open set, then X has a partition as above iff X admits a fixed-point free involution.

1. Introduction.

All spaces under discussion are separable metric.

While discussing the paper [6] with Professor A. V. Arhangel'skiĭ, the following question was raised: *does there exist a homogeneous space which can be partitioned into two homeomorphic rigid subsets*? (a space is called *rigid* if the identity is the only autohomeomorphism). In this note we will answer this question in the affirmative.

THEOREM 1.1. *Let X be a topologically complete, dense in itself space. If X admits a fixed-point free involution, then X can be partitioned into two homeomorphic rigid sets which in addition do not contain any uncountable compact subset.*

(An *involution* of a space X is an autohomeomorphism $h: X \to X$ with $h^2 = \mathrm{id}$.)

One might think that an involution having no fixed-points is a rather strange hypothesis to obtain such a decomposition. This is not true, as the following result shows.

THEOREM 1.2. *Let X be a Peano continuum with the property that no countable subset of X separates some nonempty open connected subset of X. Then the following statements are equivalent:*

(a) *X admits a fixed-point free involution,*

(b) *X can be partitioned into two homeomorphic rigid subsets which do not contain any uncountable compact subset.*

Our construction is inspired by a method originally due to Kuratowski [4] which was later rediscovered by de Groot ([2]).

An unpleasant aspect of our construction is that we use transfinite induction and the Kuratowski–Zorn Lemma. For this reason we will also include an explicit construction of a partition of $S^1 \times \boldsymbol{R}$ into homeomorphic rigid subsets.

2. Preliminaries. A cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. $\mathfrak{c}$ denotes $2^{\aleph_0}$.

The following classical result, due to Lavrentieff [6], will be important in our construction.

LEMMA 2.1. *Let X and Y be topologically complete. If $A \subset X$ and $B \subset Y$ and if $h: A \to B$ is a homeomorphism, then there are G_δ-subsets $A' \subset X$ and $B' \subset Y$ such that $A \subset A'$ and $B \subset B'$ while moreover h can be extended to a homeomorphism $h': A' \to B'$.*

The domain and range of a function f will be denoted by $\operatorname{dom}(f)$ and $\operatorname{range}(f)$, respectively. Let X be any (separable metric) space. Observe that the collection

$$\mathscr{F} = \{f\colon \operatorname{dom}(f) \text{ and } \operatorname{range}(f) \text{ are } G_\delta\text{-subsets of } X \text{ and } f\colon \operatorname{dom}(f) \to \operatorname{range}(f) \text{ is a homeomorphism}\}$$

has cardinality at most $\mathfrak{c}$.

3. Proof of Theorem 1.1. In this section we will give a proof of Theorem 1.1. To this end, let X be a topologically complete, dense in itself space and let $\mathscr{F}$ be as in Section 2. Put

$$\mathscr{G} = \{f \in \mathscr{F}\colon \exists D \subset \operatorname{dom}(f), |D| = \mathfrak{c} \text{ and } f(D) \cap D = \emptyset\}.$$

Since $|\mathscr{G}| \leqslant \mathfrak{c}$, we can enumerate $\mathscr{G}$ by $\{f_\alpha\colon \alpha < \mathfrak{c}\}$. Let h be a fixed-point free involution of X. By transfinite induction, for every $\alpha < \mathfrak{c}$, we will construct a point $a_\alpha \in \operatorname{dom}(f_\alpha)$ such that

$$a_\alpha \notin \{f_\beta(a_\beta)\colon \beta \leqslant \alpha\} \cup \{h(a_\beta)\colon \beta \leqslant \alpha\}, \tag{1}$$

$$f_\alpha(a_\alpha) \notin \{a_\beta\colon \beta \leqslant \alpha\} \cup \{h(f_\beta(a_\beta))\colon \beta \leqslant \alpha\}. \tag{2}$$

This construction is a triviality. Suppose that we have constructed the points a_β for all $\beta < \alpha$. By assumption, there is a subset $D \subset \operatorname{dom}(f_\alpha)$ of cardinality $\mathfrak{c}$ such that $D \cap f_\alpha(D) = \emptyset$. Since $|\alpha| < \mathfrak{c}$, we can therefore find a subset $D_0 \subset D$ which is also of cardinality $\mathfrak{c}$ and which misses $\{f_\beta(a_\beta)\colon \beta < \alpha\} \cup \{h(a_\beta)\colon \beta < \alpha\}$. Since f_α is one-to-one, by the same argument, we can find a subset $D_1 \subset D_0$ of cardinality $\mathfrak{c}$ such that $f_\alpha(D_1) \cap (\{a_\beta\colon \beta < \alpha\} \cup \{h(f_\beta(a_\beta))\colon \beta < \alpha\}) = \emptyset$. Take any point $x \in D_1$ and define $a_\alpha = x$. Since h has no fixed-points, a_α is clearly as required. This completes the induction.

Now put

$$F = \{a_\alpha\colon \alpha < \mathfrak{c}\} \cup \{h(f_\alpha(a_\alpha))\colon \alpha < \mathfrak{c}\}.$$

In addition, let

$$G = X \setminus (F \cup h(F)),$$

and let $G' \subset G$ be a set with the property that for any $x \in G$ we have $|G' \cap \{x, h(x)\}| = 1$. The existence of G' easily follows from the Kuratowski–Zorn Lemma. Let $A = F \cup G'$ and $B = X \setminus A$.

LEMMA 3.1. *If $x \in X$ then $|A \cap \{x, h(x)\}| = 1$.*

Proof. It clearly suffices to show that $F \cap h(F) = \emptyset$. This easily follows from (1) and (2) and from the fact that h is an involution. □

COROLLARY 3.2. *A is homeomorphic to B.*

Proof. Clearly, $h(A) = B$ and $h(B) = A$. □

LEMMA 3.3. *If $K \subset X$ is a Cantor set, then $K \cap A \neq \emptyset$ and $K \cap B \neq \emptyset$.*

Proof. Let K_0 and K_1 be disjoint Cantor sets in K and let $f\colon K_0 \to K_1$ be any homeomorphism. Then, $f \in \mathcal{G}$ and consequently, by construction, $A \cap \operatorname{dom}(f)$ is nonempty. Therefore, A intersects K_0. By (1) and (2), $f(K_0)$ intersects B. We conclude that also $B \cap K \neq \emptyset$. □

Since any uncountable compactum contains a Cantor set, the following is immediate.

COROLLARY 3.4. *A and B do not contain uncountable compact subsets. As a consequence, both A and B are dense in X.*

LEMMA 3.5. *A is rigid.*

Proof. Suppose, to the contrary, that $h\colon A \to A$ is a homeomorphism which is not the identity. By Lemma 2.1 we can find G_δ-subsets S and T in X which both contain A while in addition, h can be extended to a homeomorphism $\bar{h}\colon S \to T$. It is clear that there is an $x \in S$ such that $\bar{h}(x) \neq x$. Let C be a closed neighborhood of x in S such that $C \cap \bar{h}(C) = \emptyset$. By Corollary 3.4, A is dense in X and therefore, so is S. We conclude that S is topologically complete, being a G_δ-subset of X, and dense in itself, being dense in X. Therefore, C must have cardinality $\mathfrak{c}$. Define $f = \bar{h}|C$ and notice that $f \in \mathcal{G}$, say, $f = f_\alpha$. Observe that $a_\alpha \in A \cap C$ and that, by (1) and (2) and by the definition of A, $f_\alpha(a_\alpha) \notin A$. This contradicts the fact that $\bar{h}$ extends h. □

4. Proof of Theorem 1.2. Using an idea in Curtis and van Mill [1], in this section we will give a surprisingly simple proof of Theorem 1.2.

To this end, let X be a Peano continuum with the property that no countable subset of X separates some nonempty open subset of X. Suppose that $\{A, B\}$ is a partition into rigid homeomorphic subsets of X which do not contain uncountable compact subsets.

LEMMA 4.1. *Let $U \subset X$ be nonempty, connected and open. Then $U \cap A$ is connected.*

Proof. Suppose not. Let U_0, U_1 be a partition of $U \cap A$ consisting of nonempty open (in A) subsets of A. There are open subsets U_0' and U_1' in X which are contained in U so that $U_i' \cap A = U_i$ for $i = 0, 1$. Let $K = U \setminus (U_0' \cup U_1')$. Observe that K separates U and that K is contained in B. Since K is clearly σ-compact, it has to be countable, since B does not contain any uncountable compact subset. We conclude that some countable set separates U, which contradicts our assumptions on X. □

Now let h: $A \to B$ be any homeomorphism. We claim that h can be extended to a homeomorphism $\bar{h}$: $X \to X$. Take $x \in B$ and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of connected open neighborhoods of x in X such that for all n,

$$\text{(1)} \qquad \operatorname{diam}(U_n) < 1/n,$$

$$\text{(2)} \qquad U_{n+1}^- \subset U_n.$$

By Lemma 4.1, $U_n \cap A$ is connected for all n, and consequently, so is $h(U_n \cap A)$. Let $C = \bigcap_{n=1}^{\infty} h(U_n \cap A)^-$ (the closure is taken in X). Observe that C is a decreasing intersection of continua, hence must be a continuum itself, which obviously is contained in $X \setminus h(A) = X \setminus B = A$. Since A, by assumption, cannot contain nontrivial continua, C must contain precisely one point. Let this point be denoted by f_x. Then the function f: $X \to X$ defined by

$$f(x) = \begin{cases} f_x, & \text{if } x \in B, \\ h(x), & \text{if } x \in A, \end{cases}$$

obviously defines a continuous extension of h. In the same way we can define a continuous extension g: $X \to X$ of h^{-1}. We conclude that h can be extended to a homeomorphism $\bar{h}$: $X \to X$. First, observe that $\bar{h}$ has no fixed-points, since $\bar{h}(A) = B$. Second, $\bar{h}$ is clearly an involution, since $\bar{h}^2 | A$ is an autohomeomorphism of A, and hence, by rigidity, must be the identity on A. Since moreover A is dense, we conclude that $\bar{h}^2$ must be the identity itself.

5. A partition of $S^1 \times \boldsymbol{R}$. In this section we will show that there is an explicit construction of a partition of the ordinary cylinder $S^1 \times \boldsymbol{R}$ into two homeomorphic rigid sets.

For a coordinate system on S^1 we use the reals modulo 2π. We have a fixed-point free involution h on $S^1 \times \boldsymbol{R}$ if we define:

$$h(s, r) = (s + \pi, r) \quad \text{for all } 0 \leqslant s < \pi,$$

and

$$h(s, r) = (s-\pi, r) \quad \text{for all } \pi \leqslant s < 2\pi.$$

Put

$$A_0 = \{(s, r)\colon\ 0 \leqslant s < \pi;\ r \in \boldsymbol{R}\}$$

and

$$B_0 = \{(s, r)\colon\ \pi \leqslant s < 2\pi;\ r \in \boldsymbol{R}\},$$

respectively.

We subdivide B_0 into a countable dense union B_f of treelike spaces (= a connected subspace of a dendron) and a dense G_δ set B_g. If we put $h(B_f) = A_f$ and $h(B_g) = A_g$, then obviously, $A = A_g \cup B_f$ is homeomorphic to $B = B_g \cup A_f$.

For the construction of B_f and B_g we adapt the techniques of de Groot and Wille [3], and so we obtain rigidity of both B_g and B_f.

Let $\{\varphi_i\}_{i=1}^{\infty}$ be a countable collection of disjoint directions in the plane, such that for every $i \in N$ and every straight line l in the direction φ_i there is at most one point on l with both coordinates rational.

We construct B_f as the union of a countable disjoint collection of open straight line segments $\{H_i\}_{i=1}^{\infty}$ such that for all i, H_i has direction φ_i. On every segment H_i we will define a countable dense set D_i. Maximal connected unions of H_i's will be treelike spaces T_m without endpoints.

Let D_0 be the set of all points $(s, r) \in B_0$ such that both s and r are rational. The union $\bigcup\limits_{i=0}^{\infty} D_i$ will be called $D^{\sim}$. We will index $D^{\sim}$ according to the conventions:

$$D^{\sim} = \{d_n\colon\ n \in N,\ n \geqslant 1\},$$

and

$$\text{if } n = k\,2^l \text{ then } d_n \text{ is the } k\text{th element of } D_l.$$

We proceed by induction on the index n of $D^{\sim}$.

Start. $d_1 \in D_0$. Let $C_1 \subset B_0$ be a closed circle with midpoint d_1 and radius $r_1 \leqslant 1$. Define the two sets H_1 and H_2 to be the open segments of length r_1 which emerge from d_1 in the direction φ_1, respectively φ_2. Let S_1, respectively S_2, be two closed circle sectors of C_1 such that $H_1 \subset \operatorname{Int} S_1$, $H_2 \subset \operatorname{Int} S_2$, $S_1 \cap S_2 = \{d_1\}$ and $Q \times Q \cap \partial S_i = \{d_1\}$ where Q denotes the set of rationals and ∂ the boundary operator. Define D_1 to be a countable dense subset of H_1 and D_2 to be a countable dense subset of H_2.

Step. Suppose that C_m, d_m, H_i, S_i and D_i are defined for each $1 \leqslant m \leqslant n$ and $i < \frac{1}{2}n(n+1)$. If $n = k\,2^l$ then $l < \frac{1}{2}n(n+1)$ and hence d_n is well-defined. We can define a closed circle C_n with midpoint d_n which satisfies the following properties:

(1) $C_n \cap H_i = \emptyset$ for $i \neq l$ and $i < \frac{1}{2}n(n+1)$,

(2) for all $i < \frac{1}{2}n\,(n+1)$, if $d_n \in S_i$ then $C_n \subset \operatorname{Int} S_i$,

(3) for all $i < \frac{1}{2}n(n+1)$, if $d_n \notin S_i$ then $C_n \cap S_i = \emptyset$,

(4) the radius r_n of C_n satisfies $r_n \leqslant n^{-2}$ and $C_n \subset B_0$.

Next we define the $n+1$ open segments H_i of length r_n in the direction φ_i for $\frac{1}{2}n(n+1) \leqslant i \leqslant \frac{1}{2}n(n+3)$. On each of those H_i we chose a countable dense set D_i, and around each H_i we define a closed circle sector S_i of C_n with the following properties:

(5) for $\frac{1}{2}n(n+1) \leqslant i < j \leqslant \frac{1}{2}n(n+3)$ we have $S_i \cap S_j = \{d_n\}$,

(6) for $\frac{1}{2}n(n+1) \leqslant i \leqslant \frac{1}{2}n(n+3)$ we have $\partial S_i \cap (Q \times Q) \subset \{d_n\}$,

(7) for $j < \frac{1}{2}n(n+1) \leqslant i \leqslant \frac{1}{2}n(n+3)$ we have $H_j \cap S_i \subset \{d_n\}$,

(8) for $\frac{1}{2}n(n+1) \leqslant i \leqslant \frac{1}{2}n(n+3)$ we have $H_i \subset \operatorname{Int} S_i$.

Notice that if $n = k\,2^l$ and $l \geqslant 1$ then H_i is a branch of some treelike space which emerges from H_l. Define

$$N^{\sim} = \{m \in N\colon \exists l \in N\colon l(2l-1) \leqslant m \leqslant (l+1)(2l-1)\},$$

i.e., $m \in N^{\sim}$ if and only if H_m emerges from a point of D_0.

Next we take B_f to be the union of all the H_i's. Then B_f can be seen as the union of a countable discrete collection $\{T_m\colon m \in N^{\sim}\}$ of treelike spaces, in which T_m denotes the treelike space with initial branch H_m.

From induction conditions (2) and (8) it follows that $T_m \subset S_m$; from (2), (3), (6) and (7) we obtain that $\partial S_m \cap B_f = \emptyset$ and hence $S_m \cap B_f$ is a clopen subset of B_f for every $m \in N^{\sim}$.

(a) $T_m = B_f \cap S_m \cap \bigcap \{B_0 \backslash S_k\colon k \in N^{\sim}, k > m\}$.

It follows immediately that T_m is a component of B_f.

(b) The diameter of T_m tends to 0 if $m \to \infty$, and so does the diameter of an arbitrary sector S_n.

(c) The set $D^{\sim} \backslash D_0$ is dense in B_f and each d_n in T_m is a cutpoint of T_m which cuts T_m into $n+3$ disjoint subcomponents.

It follows immediately that B_f is rigid. Moreover, if we consider $B_f \cup A_g = B_f \cup h(B_g)$ then it is clear that no homeomorphism of $B_f \cup A_g$ can send a point of B_f to a point of A_g because B_f is a first category space and A_g is a Baire space. So we only have to show that $B_g = h(A_g)$ is rigid. To this end we make the following observation.

(d) No point outside D_0 is in the closure of more than one T_m. This follows directly from the induction conditions (3) and (5), since each T_m is contained in the interior of its S_m and its circle C_k. In the same way it follows that even subcomponents of $T_m \setminus \{d_k\}$ cannot have boundary points in common.

We will show the rigidity of B_g by proving that within every neighborhood U_p of $p \in B_g$ there exists a connected neighborhood V_p of p such that even $V_p \setminus \{p\}$ is connected for every $p \in B_g \setminus D_0$. For $d_n \in D_0$ we have that if V_d is a connected neighborhood with diameter less than r_n then $V_d \setminus \{d_n\}$ consists of precisely $n+1$ components. Since D_0 is dense, the rigidity of B_g follows.

Let $F = (B_g \setminus D_0) \cap (\bigcup_{m \in N^\sim} \mathrm{cl}(T_m))$, i.e., the points in the closures of single trees. Let $G = B_g \setminus (D_0 \cup F)$. We show our claims independently for $p \in G$, for $p \in F$ and for $p \in D_0$.

Case (i). $p \in G$. Let U be an open circle neighborhood of p with midpoint p and radius ε. The diameter of T_m is larger than $\frac{1}{2}\varepsilon$ for a finite subcollection $M^\sim$ of $N^\sim$ and so only a finite number of nowhere dense sets T_m intersect both the boundary of U and the inner circle $U^\sim = \{x\colon \varrho(x, p) < \frac{1}{2}\varepsilon\}$, where ϱ is the ordinary distance function. We obtain a neighborhood of p if we consider the component of p in $W_p = U \setminus (\bigcup_{m \in M^\sim} T_m)$. If W_p does not contain a connected neighborhood of p in B_g then there is a clopen subset K_p containing p in $W_p \cap B_g$. The boundary of K_p with respect to B_0 must then contain a closed noncontractible continuum which misses the boundary of U. This is impossible since the components of B_f are treelike and do not contain noncontractible subcontinua.

Case (ii) $p \in F$. This case is similar to the previous one but now p is not in the interior of W_p. Also the connected neighborhood will not fall apart by deleting p since the only subcontinua of $B_f \cup \{p\}$ containing p are treelike, since observation (d) guarantees that the components of $B_f \cup \{p\}$ are treelike.

Case (iii). $p \in D_0$, say $p = d_n$. In this case a sufficiently small neighborhood of p is subdivided into open sectors by the H_m emerging from p. Instead of a single set W_p we now take the component of p in $U \setminus \bigcup \{T_m\colon \frac{1}{2}n(n+1) \leqslant m \leqslant \frac{1}{2}n(n+3)\}$. This set is a basic open neighborhood of p which falls apart into $n+1$ parts when p is removed.

Since each $p \in D_0$ can only be mapped onto itself by a homeomorphism of B_g onto B_g and since D_0 is dense, the rigidity of B_g follows.

Therefore, $A_g = h(B_g)$ is rigid and also $B_f \cup A_g$ is rigid. This shows all the required properties of the example.

6. Remarks. The results in this note do not imply that spaces such as the real line $\boldsymbol{R}$, the closed unit interval I, or I^2 can be partitioned into two dense homeomorphic rigid subsets (**P 1286**). It would be interesting to know whether this is possible or not, especially for the real line. Observe that Theorem 1.2 shows that for I^2 a method such as in Section 3 does not work. We don't know whether a geometric argument does the job.

Added in proof. Independently, S. Shelah and F. van Engelen have shown that the real line $\boldsymbol{R}$ can be partitioned into homeomorphic rigid parts.

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