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ON THE EXISTENCE OF CERTAIN TYPES OF RIEMANNIAN METRICS

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1. Introduction. Let M be an n-dimensional Riemannian manifold with a (not necessarily definite) metric g.

The manifold M is said to be *locally symmetric* if its curvature tensor

$$R_{hijk} = \frac{1}{2}(g_{hk.ij}+g_{ij.hk}-g_{hj.ik}-g_{ik.hj})+g_{pq}\begin{Bmatrix} p \\ hk \end{Bmatrix}\begin{Bmatrix} q \\ ij \end{Bmatrix}-g_{pq}\begin{Bmatrix} p \\ hj \end{Bmatrix}\begin{Bmatrix} q \\ ik \end{Bmatrix}$$

satisfies the condition $R_{hijk,l} = 0$, where the dot denotes partial differentiation with respect to coordinates and the comma denotes covariant differentiation with respect to the metric g.

The manifold M is said to be of *recurrent curvature* if it satisfies the condition

$$R_{hijk,l} = c_l R_{hijk}, \quad \text{where } c_{l,m} = c_{m,l},$$

for some vector field c_l (see [5]).

As a generalization of the concept of a recurrent manifold, Lichnerowicz initiated investigations of Riemannian manifolds whose curvature tensors satisfy a relation of the form

$$R_{hijk,lm} = a_{lm} R_{hijk}, \quad \text{where } a_{lm} = a_{ml}.$$

Manifolds of this type, i.e. satisfying the above relation for some tensor a_{ij}, are called *second order recurrent* or, briefly, *2-recurrent*. The existence of 2-recurrent manifolds which are not recurrent was proved by McLenaghan and Thompson in [2].

We assume that all manifolds considered below are of dimension $n \geqslant 4$.

A Riemannian manifold M is called *conformally symmetric* if its Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \\ + \frac{R}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj})$$

satisfies the condition $C_{hijk,l} = 0$.

Clearly, the class of conformally symmetric manifolds contains all conformally flat as well as all locally symmetric manifolds. A conformally symmetric manifold is said to be *essentially conformally symmetric* if it is neither conformally flat nor locally symmetric. The existence of essentially conformally symmetric manifolds was proved by Roter [3].

Adati and Miyazawa [1] introduced the concept of a *conformally recurrent manifold*, i.e., a manifold whose Weyl conformal curvature tensor satisfies the condition $C_{hijk,l} = d_l C_{hijk}$ for some vector field d_l. All known examples of such manifolds satisfy

$$d_{l,m} = d_{m,l}, \tag{1}$$

i.e., the vector d_l may be chosen to be a local gradient. Clearly, every recurrent manifold as well as every conformally symmetric manifold is conformally recurrent.

The existence of *essentially conformally recurrent manifolds*, i.e. conformally recurrent manifolds which are neither recurrent nor conformally symmetric, was established by Roter [4].

It is easy to see that each conformally recurrent manifold satisfies $C_{hijk,lm} = b_{lm}C_{hijk}$ for some tensor field b_{lm}. Manifolds of this type are called *conformally 2-recurrent*.

Every 2-recurrent manifold satisfies $R_{hijk,lm} = R_{hijk,ml}$. It is unknown whether an analogous relation, i.e.,

$$C_{hijk,lm} = C_{hijk,ml}, \tag{2}$$

is true for conformally 2-recurrent manifolds.

In this paper we consider only conformally 2-recurrent manifolds satisfying (2), i.e., such manifolds for which the tensor b_{ij} is symmetric. It is clear that every 2-recurrent as well as every conformally recurrent manifold satisfying (1) is conformally 2-recurrent and satisfies (2). In this paper we shall prove, among others, the existence of *essentially conformally 2-recurrent manifolds*, i.e., conformally 2-recurrent manifolds which are neither 2-recurrent nor conformally recurrent.

The inclusion relations among the classes of manifolds discussed above are illustrated by the following diagram:

$$\begin{tikzcd}
D & \boxed{\begin{array}{c}\text{manifolds satisfying}\\ R_{hijk,lm} = R_{hijk,ml}\end{array}} \arrow[r] & \boxed{\begin{array}{c}\text{manifolds satisfying}\\ C_{hijk,lm} = C_{hijk,ml}\end{array}} & H \\
C & \boxed{\text{2-recurrent}} \arrow[u] \arrow[r] & \boxed{\begin{array}{c}\text{conformally 2-recurrent}\\ \text{satisfying (2)}\end{array}} \arrow[u] & G \\
B & \boxed{\text{recurrent}} \arrow[u] \arrow[r] & \boxed{\begin{array}{c}\text{conformally recurrent}\\ \text{satisfying (1)}\end{array}} \arrow[u] & F \\
A & \boxed{\text{locally symmetric}} \arrow[u] \arrow[r] & \boxed{\text{conformally symmetric}} \arrow[u] & E
\end{tikzcd}$$

As we have mentioned, McLenaghan and Thompson gave an example of a 2-recurrent manifold which is not recurrent. In this paper we shall describe a class of metrics showing all the inclusions in the above diagram, except possibly for $D \subset H$, are strict. Of particular interest will be the examples showing that $H \neq G$, $D \neq C$ and the example of an essentially conformally 2-recurrent manifold.

2. Preliminary results. In what follows each Latin index runs over $1, 2, \ldots, n$, and each Greek index over $2, 3, \ldots, n-1$.

We define the metric g in R^n, $n \geqslant 4$, by the formula

$$ds^2 = Q(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{3}$$

where $[k_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constants, and Q is independent of x^n.

The only components of Christoffel symbols, R_{hijk}, R_{ij}, C_{hijk}, $R_{hijk,l}$, $R_{ij,l}$, $C_{hijk,l}$, not identically zero are those related to (see [3])

$$\begin{Bmatrix} \lambda \\ 11 \end{Bmatrix} = -\frac{1}{2}k^{\lambda\omega}Q_{.\omega}, \quad \begin{Bmatrix} n \\ 11 \end{Bmatrix} = \frac{1}{2}Q_{.1}, \quad \begin{Bmatrix} n \\ 1\gamma \end{Bmatrix} = \frac{1}{2}Q_{.\gamma},$$

$$R_{1\lambda\mu 1} = \frac{1}{2}Q_{.\lambda\mu}, \quad R_{11} = \frac{1}{2}k^{\beta\omega}Q_{.\beta\omega},$$

$$C_{1\lambda\mu 1} = \frac{1}{2}Q_{.\lambda\mu} - \frac{1}{2(n-2)}k_{\lambda\mu}(k^{\beta\omega}Q_{.\beta\omega}),$$

$$
(4)\qquad
\begin{aligned}
R_{1\lambda\mu 1,1} &= \frac{1}{2} Q_{.\lambda\mu 1}, \quad R_{1\lambda\mu 1,\gamma} = \frac{1}{2} Q_{.\lambda\mu\gamma},\\
R_{11,1} &= \frac{1}{2} k^{\beta\omega} Q_{.\beta\omega 1}, \quad R_{11,\gamma} = \frac{1}{2} k^{\beta\omega} Q_{.\beta\omega\gamma},\\
C_{1\lambda\mu 1,1} &= \frac{1}{2} Q_{.\lambda\mu 1} - \frac{1}{2(n-2)} k_{\lambda\mu}(k^{\beta\omega} Q_{.\beta\omega 1}),\\
C_{1\lambda\mu 1,\gamma} &= \frac{1}{2} Q_{.\lambda\mu\gamma} - \frac{1}{2(n-2)} k_{\lambda\mu}(k^{\beta\omega} Q_{.\beta\omega\gamma}),
\end{aligned}
$$

where $[k^{\lambda\mu}] = [k_{\lambda\mu}]^{-1}$.

Similarly, by an elementary but somewhat lengthy calculation we can easily show that the only components of $R_{hijk,lm}$, $R_{ij,lm}$, and $C_{hijk,lm}$ not identically zero are those related to

$$
(5)\qquad
\begin{aligned}
R_{1\lambda\mu 1,11} &= \frac{1}{2} Q_{.\lambda\mu 11} + \frac{1}{4} Q_{.\lambda\mu\omega} k^{\omega\beta} Q_{.\beta}, \quad R_{1\lambda\mu 1,1\gamma} = R_{1\lambda\mu 1,\gamma 1} = \frac{1}{2} Q_{.\lambda\mu 1\gamma},\\
R_{1\lambda\mu 1,\gamma\delta} &= \frac{1}{2} Q_{.\lambda\mu\gamma\delta}, \quad R_{11,11} = \frac{1}{2} k^{\alpha\omega} Q_{.\alpha\omega 11} + \frac{1}{4} k^{\alpha\omega} Q_{.\alpha\omega\gamma} k^{\gamma\beta} Q_{.\beta},\\
R_{11,1\gamma} &= R_{11,\gamma 1} = \frac{1}{2} k^{\alpha\omega} Q_{.\alpha\omega 1\gamma}, \quad R_{11,\gamma\delta} = \frac{1}{2} k^{\alpha\omega} Q_{.\alpha\omega\gamma\delta},\\
C_{1\lambda\mu 1,11} &= \frac{1}{2} Q_{.\lambda\mu 11} + \frac{1}{4} Q_{.\lambda\mu\omega} k^{\omega\beta} Q_{.\beta} -\\
&\quad - \frac{1}{2(n-2)} k_{\lambda\mu}\left(k^{\alpha\omega} Q_{.\alpha\omega 11} + \frac{1}{2} k^{\alpha\omega} Q_{.\alpha\omega\gamma} k^{\gamma\beta} Q_{.\beta}\right),\\
C_{1\lambda\mu 1,1\delta} &= C_{1\lambda\mu 1,\delta 1} = \frac{1}{2} Q_{.\lambda\mu 1\delta} - \frac{1}{2(n-2)} k_{\lambda\mu}(k^{\alpha\omega} Q_{.\alpha\omega 1\delta}),\\
C_{1\lambda\mu 1,\gamma\delta} &= \frac{1}{2} Q_{.\lambda\mu\gamma\delta} - \frac{1}{2(n-2)} k_{\lambda\mu}(k^{\alpha\omega} Q_{.\alpha\omega\gamma\delta}).
\end{aligned}
$$

3. Main results. Let M be the n-dimensional manifold ($n = 2m$) with the metric given by (3) and let

$$
(6)\qquad Q(x^1, \ldots, x^{n-1})
$$

$$
= 4\left\{h(x^1)[v_{\alpha\beta} x^\alpha x^\beta \cos f(x^1) + w_{\alpha\beta} x^\alpha x^\beta \sin f(x^1)] - \frac{1}{2} k_{\alpha\beta} x^\alpha x^\beta p(x^1)\right\},
$$

where f, h, and p are functions of x^1 only and the matrices $[k_{\alpha\beta}]$, $[v_{\alpha\beta}]$, and $[w_{\alpha\beta}]$ are of the form

$$(7)\qquad \begin{gathered}[k_{\alpha\beta}] = \begin{bmatrix} -2 & & & \\ & -2 & 0 & \\ & & \ddots & \\ 0 & & & -2 \end{bmatrix}, \quad [v_{\alpha\beta}] = \begin{bmatrix} 1 & & & & \\ & -1 & & 0 & \\ & & \ddots & & \\ 0 & & & 1 & \\ & & & & -1 \end{bmatrix}, \\ [w_{\alpha\beta}] = \begin{bmatrix} W & & & \\ & W & 0 & \\ & 0 & \ddots & \\ & & & W \end{bmatrix}, \quad \text{where } W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.\end{gathered}$$

For $n = 4$ this metric coincides, modulo a coordinate change, with the metric of McLenaghan and Thompson [2].

One can easily show that

$$k_{\alpha\beta}k^{\alpha\beta} = 2(m-1) = n-2, \quad v_{\alpha\beta}k^{\alpha\beta} = 0, \quad w_{\alpha\beta}k^{\alpha\beta} = 0,$$

$$Q_{.\alpha\beta} = 8\left[h(v_{\alpha\beta}\cos f + w_{\alpha\beta}\sin f) - \frac{1}{2}k_{\alpha\beta}p\right], \quad Q_{.\alpha\beta\gamma} = 0,$$

$$k^{\alpha\beta}Q_{.\alpha\beta} = -8(m-1)p.$$

Using the above equations and (4), (5) one can see that the only components of R_{hijk}, R_{ij}, C_{hijk}, $R_{hijk,l}$, $R_{ij,l}$, $C_{hijk,l}$, $R_{hijk,lm}$, $R_{ij,lm}$, $C_{hijk,lm}$ not identically zero are those related to

$$R_{1aa1} = \begin{cases} 4(h\cos f + p) & \text{for even } a, \\ 4(-h\cos f + p) & \text{for odd } a, \end{cases}$$

$$R_{1a(a+1)1} = -4h\sin f \quad \text{for even } a,$$

$$R_{11} = -4(m-1)p,$$

$$C_{1aa1} = \begin{cases} 4h\cos f & \text{for even } a, \\ -4h\cos f & \text{for odd } a, \end{cases}$$

$$C_{1a(a+1)1} = -4h\sin f \quad \text{for even } a,$$

$$R_{11,1} = -4(m-1)p' = R'_{11}, \quad \text{where } p' = dp/dx^1,$$

$$R_{1aa1,1} = R'_{1aa1},$$

$$R_{1a(a+1)1,1} = R'_{1a(a+1)1} \quad \text{for even } a,$$

$$C_{1aa1,1} = C'_{1aa1},$$

$$C_{1a(a+1)1,1} = C'_{1a(a+1)1} \quad \text{for even } a,$$

$$R_{11,11} = -4(m-1)p'' = R''_{11}, \quad \text{where } p'' = d^2p/(dx^1)^2,$$

$$R_{1a(a+1)1,11} = R''_{1a(a+1)1} \quad \text{for even } a,$$

$$R_{1aa1,11} = R''_{1aa1}, \quad C_{1aa1,11} = C''_{1aa1},$$

$$C_{1a(a+1)1,11} = C''_{1a(a+1)1} \quad \text{for even } a.$$

In this section M denotes the manifold R^n with the metric defined by (3), (6), and (7). From the above formulas we obtain

LEMMA 1. *The manifold M has the following properties*: $R = 0$, $R_{ij,k} = R_{ik,j}$, $R_{hijk,lm} = R_{hijk,ml}$, $C_{hijk,lm} = C_{hijk,ml}$.

LEMMA 2. (a) *M is locally symmetric if and only if the functions h and p are constant and $hf' = 0$.*

(b) *M is conformally flat if and only if $h = 0$.*

(c) *M is conformally symmetric if and only if the function h is constant and $hf' = 0$.*

In what follows we assume that $h \neq 0$ everywhere.

LEMMA 3. *The manifold M is recurrent if and only if $f =$ const and $p' = ph'/h$.*

Proof. Assume that $R_{hijk,l} = c_l R_{hijk}$. Then

$$R_{1aa1,1} = c_1 R_{1aa1}, \quad R_{1a(a+1)1,1} = c_1 R_{1a(a+1)1}$$

and the following system of equations is satisfied:

$$\begin{aligned} h'\cos f - hf'\sin f + p' &= c_1(h\cos f + p), \\ -h'\cos f + hf'\sin f + p' &= c_1(-h\cos f + p), \\ -h'\sin f - hf'\cos f &= -c_1 h\sin f. \end{aligned} \tag{8}$$

It follows now immediately that $f' = 0$ and $p' = ph'/h$.

Conversely, assume that $f =$ const and $p' = ph'/h$. Then (8) is satisfied by $c_1 = h'/h$. Taking $c_i = 0$ for $i = 2, \ldots, n$, we obtain $R_{hijk,l} = c_l R_{hijk}$ and $c_{l,m} = c_{m,l}$.

The proofs of the following lemmas are analogous.

LEMMA 4. *The manifold M is 2-recurrent if and only if*

$$h^2 f' = \text{const} \quad \textit{and} \quad p'' = \frac{p}{h}\left(h'' - h(f')^2\right).$$

LEMMA 5. *The manifold M is conformally recurrent if and only if* $f = \text{const}$.

LEMMA 6. *The manifold M is conformally 2-recurrent if and only if* $h^2 f' = \text{const}$.

THEOREM 1. *There exist essentially conformally 2-recurrent Riemannian manifolds.*

Proof. By Lemmas 2 and 4-6 it suffices to take the manifold M with the metric g given by (3), (6), (7) with $h \neq 0$ everywhere, $h^2 f' = \text{const}$, $f \neq \text{const}$, and $p'' \neq p(h'' - h(f')^2)/h$.

THEOREM 2. *There exist manifolds satisfying $R_{hijk,lm} = R_{hijk,ml}$ which are not 2-recurrent.*

Proof. From Lemmas 1 and 4 it follows that the manifold M with the metric g, given by (3), (6), (7) with $h^2 f' \neq \text{const}$, satisfies the required conditions.

Similarly, using Lemmas 1 and 6, we obtain

THEOREM 3. *There exist manifolds satisfying $C_{hijk,lm} = C_{hijk,ml}$ which are not conformally 2-recurrent.*

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