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AN INTEGRAL FORMULA FOR A RIEMANNIAN MANIFOLD WITH TWO ORTHOGONAL COMPLEMENTARY DISTRIBUTIONS

BY

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1. Introduction. The *second fundamental form* B of a distribution D on a Riemannian manifold $(M, \langle\ ,\ \rangle)$ has been defined by Reinhart [9]: Given two vector fields X and Y tangent to D, $B(X, Y)$ is the normal component of the field

$$\tfrac{1}{2}(\nabla_X Y + \nabla_Y X),$$

where ∇ is the Levi-Cività connection on M. B is a section of the bundle $\mathrm{Hom}(D \otimes D, D^\perp)$, where $D^\perp$ denotes the orthogonal complement of D. Its trace H (with respect to the Riemannian structure) is called the *mean curvature vector* of D. H is a vector field on M orthogonal to D. If D is integrable, $x \in M$, and L $(x \in L)$ is the leaf of the foliation $\mathscr{F}$ of M corresponding to D, then $B(x)$ and $H(x)$ are the second fundamental form and the mean curvature vector of L at x, respectively.

In this paper, we deal with two orthogonal distributions D_1 and D_2 on a Riemannian manifold $(M, \langle\ ,\ \rangle)$. We assume that

$$p+q = m, \quad \text{where } p = \dim D_1,\ q = \dim D_2 \text{ and } m = \dim M.$$

We consider the mean curvature vectors H_i of D_i and calculate the sum

$$\operatorname{div} H_1 + \operatorname{div} H_2.$$

Assuming M to be closed and applying Green's Theorem we get global results which generalize some known results concerning integrals of curvatures for foliated manifolds.

Throughout the paper everything (manifolds, foliations, metrics, etc.) is assumed to be C^∞-differentiable.

2. Results. Let us take a local orthonormal frame $e_1, \ldots, e_m$ adapted to D_1 and D_2, i.e., we assume that e_i is tangent to D_1 for $i = 1, \ldots, p$ and e_α is tangent to D_2 for $\alpha = p+1, \ldots, m$. Hereafter, the indices i, j, k range over the set $\{1, \ldots, p\}$ while the indices α, β, γ range over $\{p+1, \ldots, m\}$. If v is a vector tangent to M, then we write

$$v = v^{\perp} + v^{\mathsf{T}},$$

where v^{T} is tangent to D_1 and $v^{\perp}$ belongs to D_2.

The integrability tensors T_n of D_n $(n = 1, 2)$ are defined as follows:

$$T_1(X_1, Y_1) = \tfrac{1}{2}[X_1, Y_1]^{\perp}, \quad T_2(X_2, Y_2) = \tfrac{1}{2}[X_2, Y_2]^{\mathsf{T}}$$

for vector fields X_n and Y_n tangent to D_n. T_n is a section of the bundle $\mathrm{Hom}(D_n \otimes D_n, D_n)$. The distribution D_n is integrable if and only if $T_n = 0$.

Let B_n $(n = 1, 2)$ be the second fundamental forms of D_n. Then the mean curvature vectors H_n of D_n are given by

$$H_1 = \sum_i B_1(e_i, e_i) = \sum_i (\nabla_{e_i} e_i)^{\perp}$$

and

$$H_2 = \sum_\alpha B_2(e_\alpha, e_\alpha) = \sum_\alpha (\nabla_{e_\alpha} e_\alpha)^{\mathsf{T}}.$$

Therefore,

$$\begin{aligned}(1)\qquad \operatorname{div} H_1 + \operatorname{div} H_2 &= \sum_i \langle \nabla_{e_i} H_1, e_i\rangle + \sum_\alpha \langle \nabla_{e_\alpha} H_1, e_\alpha\rangle \\ &\quad + \sum_i \langle \nabla_{e_i} H_2, e_i\rangle + \sum_\alpha \langle \nabla_{e_\alpha} H_2, e_\alpha\rangle \\ &= -\langle H_1, \sum_i \nabla_{e_i} e_i\rangle - \langle H_2, \sum_\alpha \nabla_{e_\alpha} e_\alpha\rangle \\ &\quad + \sum_\alpha \langle \nabla_{e_\alpha} H_1, e_\alpha\rangle + \sum_i \langle \nabla_{e_i} H_2, e_i\rangle \\ &= -|H_1|^2 - |H_2|^2 + \sum_{i,\alpha} \big(\langle \nabla_{e_\alpha}(\nabla_{e_i} e_i)^{\perp}, e_\alpha\rangle \\ &\quad + \langle \nabla_{e_i}(\nabla_{e_\alpha} e_\alpha)^{\mathsf{T}}, e_i\rangle\big).\end{aligned}$$

Applying the definition

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

of the curvature tensor we get

$$\begin{aligned}(2)\quad & \langle \nabla_{e_\alpha}(\nabla_{e_i} e_i)^{\perp}, e_\alpha\rangle + \langle \nabla_{e_i}(\nabla_{e_\alpha} e_\alpha)^{\mathsf{T}}, e_i\rangle \\ &= 2\langle R(e_i, e_\alpha) e_\alpha, e_i\rangle + \langle \nabla_{e_i} \nabla_{e_\alpha} e_i, e_\alpha\rangle + \langle \nabla_{e_\alpha} \nabla_{e_i} e_\alpha, e_i\rangle \\ &\quad + \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha\rangle + \langle \nabla_{[e_i, e_\alpha]} e_\alpha, e_i\rangle \\ &\quad - \langle \nabla_{e_\alpha}(\nabla_{e_i} e_i)^{\mathsf{T}}, e_\alpha\rangle - \langle \nabla_{e_i}(\nabla_{e_\alpha} e_\alpha)^{\perp}, e_i\rangle.\end{aligned}$$

Moreover,

$$(3)\qquad \langle \nabla_{e_\alpha}(\nabla_{e_i} e_i)^{\mathrm{T}}, e_\alpha\rangle = -\langle \nabla_{e_i} e_i, (\nabla_{e_\alpha} e_\alpha)^{\mathrm{T}}\rangle$$

and

$$(4)\qquad \langle \nabla_{e_i}(\nabla_{e_\alpha} e_\alpha)^{\perp}, e_i\rangle = -\langle \nabla_{e_\alpha} e_\alpha, (\nabla_{e_i} e_i)^{\perp}\rangle.$$

Also,

$$(5)\qquad \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha\rangle = -\langle \nabla_{[e_\alpha, e_i]} e_\alpha, e_i\rangle = \langle \nabla_{[e_i, e_\alpha]} e_\alpha, e_i\rangle.$$

Comparing equalities (1)–(5) we obtain

$$(6)\qquad \begin{aligned} \operatorname{div} H_1 + \operatorname{div} H_2 = & -|H_1|^2 - |H_2|^2 + 2\sum_{i,\alpha} \langle R(e_i, e_\alpha) e_\alpha, e_i\rangle \\ & + \sum_i \langle \nabla_{e_i} e_i, H_2\rangle + \sum_\alpha \langle \nabla_{e_\alpha} e_\alpha, H_1\rangle \\ & + \sum_{i,\alpha} (\langle \nabla_{e_i} \nabla_{e_\alpha} e_i, e_\alpha\rangle + \langle \nabla_{e_\alpha} \nabla_{e_i} e_\alpha, e_i\rangle \\ & + 2\langle \nabla_{[e_i, e_\alpha]} e_\alpha, e_i\rangle). \end{aligned}$$

The sum

$$\sum_{i,\alpha} \langle R(e_i, e_\alpha) e_\alpha, e_i\rangle$$

does not depend on the frame $e_1, \ldots, e_m$. It depends only on the distributions D_1 and D_2, and will be denoted by $K(D_1, D_2)$. ($K(D_1, D_2)$ is a generalization of the Ricci curvature which is sometimes called the *mean curvature* of D_1 (see [10]). This term does not fit for our situation.)

Since $[e_i, e_\alpha] = \nabla_{e_i} e_\alpha - \nabla_{e_\alpha} e_i$,

$$\begin{aligned} \sum_\alpha \langle \nabla_{e_i} \nabla_{e_\alpha} e_i, e_\alpha\rangle & = \sum_\alpha e_i \langle \nabla_{e_\alpha} e_i, e_\alpha\rangle - \sum_\alpha \langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha\rangle \\ & = -e_i \langle e_i, \sum_\alpha \nabla_{e_\alpha} e_\alpha\rangle - \sum_\alpha \langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha\rangle \\ & = -e_i \langle e_i, H_2\rangle - \sum_\alpha \langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha\rangle \end{aligned}$$

and

$$\sum_i \langle \nabla_{e_\alpha} \nabla_{e_i} e_\alpha, e_i\rangle = -e_\alpha \langle e_\alpha, H_1\rangle - \sum_i \langle \nabla_{e_i} e_\alpha, \nabla_{e_\alpha} e_i\rangle,$$

we can write (6) in the form

$$(7)\quad \operatorname{div} H_1 + \operatorname{div} H_2 = -|H_1|^2 - |H_2|^2 + 2K(D_1, D_2) - \sum_i (e_i \langle e_i, H_2\rangle - \langle \nabla_{e_i} e_i, H_2\rangle) - \sum_\alpha (e_\alpha \langle e_\alpha, H_1\rangle - \langle \nabla_{e_\alpha} e_\alpha, H_1\rangle) - 2\sum_{i,\alpha} (\langle \nabla_{e_i} e_\alpha, \nabla_{e_\alpha} e_i\rangle - \langle \nabla_{\nabla_{e_i} e_\alpha} e_\alpha, e_i\rangle + \langle \nabla_{\nabla_{e_\alpha} e_i} e_\alpha, e_i\rangle).$$

Next,

$$(8)\quad e_i\langle e_i, H_2\rangle - \langle \nabla_{e_i} e_i, H_2\rangle = \langle e_i, \nabla_{e_i} H_2\rangle,$$

$$(9)\quad e_\alpha\langle e_\alpha, H_1\rangle - \langle \nabla_{e_\alpha} e_\alpha, H_1\rangle = \langle e_\alpha, \nabla_{e_\alpha} H_1\rangle$$

and

$$(10)\quad \langle \nabla_{e_i} e_\alpha, \nabla_{e_\alpha} e_i\rangle - \langle \nabla_{\nabla_{e_i} e_\alpha} e_\alpha, e_i\rangle + \langle \nabla_{\nabla_{e_\alpha} e_i} e_\alpha, e_i\rangle = \sum_j \langle \nabla_{e_i} e_\alpha, e_j\rangle \langle \nabla_{e_\alpha} e_i, e_j\rangle + \sum_\beta \langle \nabla_{e_i} e_\alpha, e_\beta\rangle \langle \nabla_{e_\alpha} e_i, e_\beta\rangle - \sum_j \langle \nabla_{e_i} e_\alpha, e_j\rangle \langle \nabla_{e_j} e_\alpha, e_i\rangle - \sum_\beta \langle \nabla_{e_i} e_\alpha, e_\beta\rangle \langle \nabla_{e_\beta} e_\alpha, e_i\rangle - \sum_j \langle \nabla_{e_\alpha} e_i, e_j\rangle \langle \nabla_{e_j} e_\alpha, e_i\rangle - \sum_\beta \langle \nabla_{e_\alpha} e_i, e_\beta\rangle \langle \nabla_{e_\beta} e_\alpha, e_i\rangle.$$

From (7)–(10) and the equalities

$$(11)\quad \sum_i \langle e_i, \nabla_{e_i} H_2\rangle = \operatorname{div} H_2 - \sum_\alpha \langle e_\alpha, \nabla_{e_\alpha} H_2\rangle = \operatorname{div} H_2 + |H_2|^2$$

and

$$(11')\quad \sum_\alpha \langle e_\alpha, \nabla_{e_\alpha} H_1\rangle = \operatorname{div} H_1 - \sum_i \langle e_i, \nabla_{e_i} H_1\rangle = \operatorname{div} H_1 + |H_1|^2$$

it follows that

$$\operatorname{div} H_1 + \operatorname{div} H_2 = -|H_1|^2 - |H_2|^2 + K(D_1, D_2) - \sum_{i,j,\alpha} (\langle \nabla_{e_i} e_\alpha, e_j\rangle \langle \nabla_{e_\alpha} e_i, e_j\rangle + \langle \nabla_{e_\alpha} e_i, e_j\rangle \langle \nabla_{e_j} e_\alpha, e_i\rangle - \langle e_\alpha, \nabla_{e_i} e_j\rangle \langle e_\alpha, \nabla_{e_j} e_i\rangle) - \sum_{i,\alpha,\beta} (\langle \nabla_{e_i} e_\alpha, e_\beta\rangle \langle \nabla_{e_\alpha} e_i, e_\beta\rangle + \langle \nabla_{e_i} e_\alpha, e_\beta\rangle \langle \nabla_{e_\beta} e_i, e_\alpha\rangle - \langle e_i, \nabla_{e_\alpha} e_\beta\rangle \langle e_i, \nabla_{e_\beta} e_\alpha\rangle),$$

i.e., that

(12) $$\operatorname{div} H_1 + \operatorname{div} H_2 = -|H_1|^2 - |H_2|^2 + K(D_1, D_2) + \sum_{\alpha,\beta} \langle (\nabla_{e_\alpha} e_\beta)^{\top}, (\nabla_{e_\beta} e_\alpha)^{\top} \rangle + \sum_{i,j} \langle (\nabla_{e_i} e_j)^{\perp}, (\nabla_{e_j} e_i)^{\perp} \rangle.$$

Finally, since

$$(\nabla_{e_i} e_j)^{\perp} + (\nabla_{e_j} e_i)^{\perp} = 2B_1(e_i, e_j)$$

and

$$(\nabla_{e_i} e_j)^{\perp} - (\nabla_{e_j} e_i)^{\perp} = 2T_1(e_i, e_j),$$

we have

(13) $$\sum_{i,j} \langle (\nabla_{e_i} e_j)^{\perp}, (\nabla_{e_j} e_i)^{\perp} \rangle = \sum_{i,j} \left(|B_1(e_i, e_j)|^2 - |T_1(e_i, e_j)|^2\right) = |B_1|^2 - |T_1|^2.$$

Similarly,

(14) $$\sum_{\alpha,\beta} \langle (\nabla_{e_\alpha} e_\beta)^{\top}, (\nabla_{e_\beta} e_\alpha)^{\top} \rangle = |B_2|^2 - |T_2|^2.$$

Equalities (12)–(14) lead to the following

PROPOSITION. *If D_1 is a distribution on a Riemannian manifold M and D_2 is the orthogonal complement of D_1, then*

(15) $$\operatorname{div} H_1 + \operatorname{div} H_2 = K(D_1, D_2) + |B_1|^2 + |B_2|^2 - |H_1|^2 - |H_2|^2 - |T_1|^2 - |T_2|^2,$$

where B_n, H_n and T_n ($n = 1, 2$) denote the second fundamental forms, mean curvature vectors and integrability tensors of distributions D_n, respectively.

The following theorem and corollaries follow immediately from the Proposition.

THEOREM 1. *If D_1 and D_2 are complementary orthogonal distributions on a closed oriented Riemannian manifold M, then*

$$\int_M \{K(D_1, D_2) + |B_1|^2 + |B_2|^2 - |H_1|^2 - |H_2|^2 - |T_1|^2 - |T_2|^2\}\, \Omega = 0,$$

where Ω is the volume form on M.

COROLLARY 1. *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are orthogonal foliations of complementary dimensions on a closed oriented Riemannian manifold M, then*

(16) $$\int_M \{K(\mathcal{F}_1, \mathcal{F}_2) + |B_1|^2 + |B_2|^2 - |H_1|^2 - |H_2|^2\}\, \Omega = 0.$$

COROLLARY 2. *If $\mathcal{F}$ is a Riemannian foliation on a closed oriented Riemannian manifold M, the Riemannian metric on M is bundle-like with* *respect to $\mathcal{F}$ and the normal bundle $\mathcal{F}^\perp$ of $\mathcal{F}$ is integrable, then*

$$\int_M \{K(\mathcal{F}, \mathcal{F}^\perp)+|B|^2-|H|^2\}\,\Omega = 0,$$

where B and H are the second fundamental form and the mean curvature vector of $\mathcal{F}$.

In fact, in this case $\mathcal{F}^\perp$ is totally geodesic, i.e., its second fundamental form vanishes identically (see, e.g., [3] or [8]).

COROLLARY 3 ([6], Theorem VII.2.1). *If $\mathcal{F}$ is a codimension-one foliation of a closed oriented Riemannian manifold M, then*

$$\int_M \{\mathrm{Ric}(N)-2k_2(\mathcal{F})\}\,\Omega = 0,$$

where N is the unit vector field on M orthogonal to $\mathcal{F}$, and $k_2(\mathcal{F})$ is the second mean curvature of (the leaves of) $\mathcal{F}$.

In fact, in this case the normal bundle $\mathcal{F}^\perp$ of $\mathcal{F}$ is integrable, $K(\mathcal{F}, \mathcal{F}^\perp) = \mathrm{Ric}(N)$, and the length $|\nabla_N N|$ of the second fundamental form of $\mathcal{F}^\perp$ is equal to the length of its mean curvature vector. The second mean curvature of $\mathcal{F}$ is equal to the sum $\sum_{i<j} \lambda_i \lambda_j$, where λ_i $(i = 1, \ldots, m-1)$ are the principal curvatures of the leaves of $\mathcal{F}$. The square of the length of the second fundamental form of $\mathcal{F}$ equals $\sum_i \lambda_i^2$ while the square of the length of its mean curvature vector amounts $(\sum_i \lambda_i)^2$.

Utilizing equality (16) once again we get the following

THEOREM 2. *If $\mathcal{F}$ is a minimal foliation of a Riemannian manifold M, the normal bundle $\mathcal{F}^\perp$ of $\mathcal{F}$ is integrable and $K(\mathcal{F}, \mathcal{F}^\perp) > 0$, then $\mathcal{F}$ has no compact leaves.*

Proof. Let us assume that L is a compact leaf of $\mathcal{F}$. The mean curvature vector $H^\perp$ of $\mathcal{F}^\perp$ is tangent to L and we may consider its divergence $\mathrm{div}_L H^\perp$ on L. Equality (11) shows that

$$\mathrm{div}_L H^\perp = \mathrm{div}\, H^\perp + |H^\perp|^2.$$

Applying the Proposition and Green's Theorem we get

$$0 = \int_L \mathrm{div}_L H^\perp = \int_L (K(\mathcal{F}, \mathcal{F}^\perp)+|B|^2+|B^\perp|^2) > 0,$$

where B and $B^\perp$ are the second fundamental forms of $\mathcal{F}$ and $\mathcal{F}^\perp$, respectively. The contradiction just obtained completes the proof.

The following corollary improves Theorem 5.3 of [11] and could be compared with Theorem 1 (ii) of [2].

COROLLARY 4. *A codimension-one minimal foliation of a Riemannian manifold of positive Ricci curvature has no closed leaves.*

3. Remarks, examples and applications.

A. Brito et al. [1] proved that if $\mathscr{F}$ is a codimension-one transversely oriented foliation of a closed $(m+1)$-dimensional manifold M of constant sectional curvature c, k_j is the j-th mean curvature of $\mathscr{F}$, then

$$\frac{1}{\operatorname{vol}(M)} \int_M k_j = \begin{cases} c^{j/2}\binom{m/2}{j/2} & \text{when } m \text{ and } j \text{ are even,} \\ 0 & \text{when either } m \text{ or } j \text{ is odd.} \end{cases}$$

In particular,

$$\frac{1}{\operatorname{vol}(M)} \int_M k_2 = \frac{1}{2} mc.$$

This equality follows immediately from our Corollary 3.

B. If M is a closed oriented surface with a one-dimensional distribution, then (16) shows that the integral $\int_M K$ of the Gaussian curvature of M (with respect to an arbitrary metric) vanishes. Applying the Gauss–Bonnet Theorem we can arrive at the following classical result: The Euler characteristic of an orientable closed surface M vanishes if M admits a one-dimensional distribution.

Similarly, if X is a vector field on a closed oriented surface M, X has only isolated singularities $x_1, \ldots, x_s$, Y is a vector field orthogonal to X on $M_0 = M \backslash \{x_1, \ldots, x_s\}$, $|Y| = 1$ and $Z = X/|X|$, then applying Corollary 1 to the distributions spanned by Y and Z we get

$$\int_{M_0} \{\operatorname{div} \nabla_Y Y + \operatorname{div} \nabla_Z Z\}\, \Omega = \int_{M_0} K\Omega = \int_M K\Omega = 2\pi\chi(M),$$

where Ω denotes the area element on M. Applying Green's Theorem to

$$M \backslash \bigcup_{h=1}^{s} D(x_h, r),$$

where $D(x_h, r)$ is the disc of radius r centered at x_h, we obtain the equality

$$\lim_{r \to 0^+} \sum_{h=1}^{s} \int_{C(x_h, r)} \{\nabla_Y Y \lrcorner\, \Omega + \nabla_Z Z \lrcorner\, \Omega\} = 2\pi\chi(M),$$

where $C(x_h, r)$ denotes the boundary of $D(x_h, r)$.

C. Corollary 3 shows that if the Ricci curvature of a closed manifold M is non-negative everywhere and strictly positive at a point, then there are no codimension-one minimal (harmonic in Kamber–Tondeur's terminology) foliations on M. This result is due to Kamber and Tondeur ([5], Theorem 1.27). Also, if $\mathcal{F}$ is a codimension-one minimal foliation of M, $\mathrm{Ric}_M \geqslant 0$, then $\mathcal{F}$ is geodesic, and the Ricci curvature of M in the direction orthogonal to M vanishes.

D. Equality (15) could be compared with Theorem 2.26 of [4], where the authors proved that

$$|A_1|^2 = |A_2|^2 + \langle \Delta\pi, \pi \rangle - K(\mathcal{F}, \mathcal{F}^\perp),$$

where $\mathcal{F}$ is a Riemannian foliation of a manifold M with a bundle-like metric $\langle\ ,\ \rangle$, $\mathcal{F}^\perp$ is the orthogonal complement of $\mathcal{F}$, A_1 — the second fundamental form of $\mathcal{F}$, A_2 — the integrability tensor of $\mathcal{F}^\perp$, π — the one form on M with values in the bundle $\mathcal{F}^\perp$, which assigns to each v of TM its orthogonal projection onto $\mathcal{F}^\perp$, and $\Delta\pi$ — the laplacian of π defined by the Levi-Cività connection on M and the unique Riemannian torsion free connection in $\mathcal{F}^\perp$. (Note that our notions of second fundamental form and integrability tensor differ from those in [4].)

E. Integrating both sides of (15) over an open subset U of M bounded by a closed leaf L of a codimension-one foliation $\mathcal{F}$ we get

$$\int_L h\omega = \int_U \left(\mathrm{Ric}(N) - 2k_2(\mathcal{F})\right)\Omega, \tag{17}$$

where we keep the notation of Corollary 3 and denote by h and ω the mean curvature function of L and the volume form of L, respectively. Equality (17) holds, e.g., when U is a Reeb component of a 2-dimensional foliation $\mathcal{F}$ of a 3-dimensional manifold. In particular, if $\mathcal{F}$ is a Reeb foliation of S^3 (with the standard metric) and the only closed leaf of $\mathcal{F}$ coincides with the Clifford torus, then

$$\int_U k_2(\mathcal{F}) = \pi^2.$$

F. An interesting example of two orthogonal foliations of complementary dimensions can be obtained as follows:

Let M be an oriented hypersurface in an oriented Riemannian manifold $\bar{M}$ of constant curvature c. Denote by A the second fundamental tensor of M and by

$$\lambda_1(x) \leqslant \lambda_2(x) \leqslant \ldots \leqslant \lambda_m(x) \qquad (x \in M,\ m = \dim M)$$

the principal curvatures of M at x. Then each λ_j is a continuous function on M.

Let us suppose that there exists a number p, $1 \leqslant p \leqslant m-1$, such that

$$\lambda_1(x) = \ldots = \lambda_p(x) < \lambda_{p+1}(x) = \ldots = \lambda_m(x) \tag{18}$$

for any x of M. Hypersurfaces satisfying (18) (actually, hypersurfaces with k

($1 \leqslant k \leqslant m$) principal curvatures of constant multiplicities) were considered by Otsuki [7], who proved that the functions $\lambda = \lambda_1$ and $\mu = \lambda_m$ are differentiable and that the corresponding distributions of eigenvectors

$$D_\lambda = \{v \in TM;\ Av = \lambda \cdot v\} \quad \text{and} \quad D_\mu = \{v \in TM;\ Av = \mu \cdot v\}$$

are integrable.

Let B_λ, B_μ, H_λ and H_μ be the second fundamental forms and the mean curvature vectors of D_λ and D_μ, respectively. If R is the curvature tensor on $\bar{M}$, N is a unit normal vector field on M, X and Y are sections of D_λ, $|X| = |Y| = 1$, and Z is a section of D_μ, then

$$\begin{aligned} 0 &= \langle R(X, Z)N, Y\rangle = \langle \nabla_X \nabla_Z N - \nabla_Z \nabla_X N - \nabla_{[X,Z]} N, Y\rangle \\ &= \langle \nabla_X(\mu Z) - \nabla_Z(\lambda X) - \langle [X, Z], Y\rangle \nabla_Y N, Y\rangle \\ &= X\mu \langle Z, Y\rangle + \mu\langle \nabla_X Z, Y\rangle - Z\lambda\langle X, Y\rangle - \lambda\langle \nabla_Z X, Y\rangle \\ &\quad - \lambda\langle \nabla_X Z, Y\rangle + \lambda\langle \nabla_Z X, Y\rangle \\ &= (\lambda-\mu)\langle B_\lambda(X, Y), Z\rangle - \langle X, Y\rangle - \langle \operatorname{grad}\lambda, Z\rangle, \end{aligned}$$

where ∇ is the Levi–Cività connection on $\bar{M}$. This shows that

$$B_\lambda(X, Y) = \frac{\langle X, Y\rangle}{\lambda-\mu} \operatorname{grad}\lambda. \tag{19}$$

Similarly, if X and Y are sections of D_λ, then

$$B_\mu(X, Y) = \frac{\langle X, Y\rangle}{\mu-\lambda} \operatorname{grad}\mu. \tag{20}$$

From (19) and (20) we get

$$|B_\lambda|^2 - |H_\lambda|^2 = (p-p^2)\frac{|\operatorname{grad}\lambda|^2}{(\lambda-\mu)^2} \tag{21}$$

and

$$|B_\mu|^2 - |H_\mu|^2 = (q-q^2)\frac{|\operatorname{grad}\mu|^2}{(\lambda-\mu)^2}, \tag{22}$$

where p and q are the multiplicities of λ and μ, respectively.

Comparing (15), (21) and (22) we get

$$\begin{aligned} \operatorname{div} H_\lambda + \operatorname{div} H_\mu &= pq(c+\lambda\mu) \\ &\quad + \frac{1}{(\lambda-\mu)^2}\{(p-p^2)|\operatorname{grad}\lambda|^2 + (q-q^2)|\operatorname{grad}\mu|^2\} \end{aligned} \tag{23}$$

since $K_M(X \wedge Z) = c + \lambda\mu$ whenever X belongs to D_λ, Z belongs to D_μ and $|X| = |Z| = 1$.

It seems to be reasonable to search for formulae analogous to (23) in the case of a hypersurface of k ($k > 2$) principal curvatures of constant multiplicities.

Added in proof. After the paper was written, the author learned that the integral formula of Theorem 1 was obtained by A. Ranjan, *Structural equations and an integral formula for foliated manifolds*, Geom. Dedicata 20 (1986), pp. 85–91, in the case of two orthogonal complementary foliations.

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