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AN INFINITE SOLITAIRE GAME WITH A RANDOM STRATEGY

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The purpose of this note is to point out a class of problems which are best handled with random, i.e., Monte Carlo, strategies.

Let $C = \{0, 1\}^{\omega}$ be the Cantor space of ω-sequences of 0's and 1's, with the usual product measure μ, i.e., for any set $A \subseteq C$ we let

$$\mu(A) = \lambda\Big\{\sum_{n=0}^{\infty} x_n/2^{n+1}\colon (x_0, x_1, \ldots) \in A\Big\},$$

where λ is the Lebesgue measure in the interval $[0, 1]$, whenever the right-hand side exists.

For any $x = (x_0, x_1, \ldots) \in C$ and $n < \omega$ we let

$$x \upharpoonright n = (x_0, \ldots, x_{n-1})$$

and, for any μ-measurable $A \subseteq C$,

$$\mu(x \upharpoonright n, A) = \mu\{y \in A\colon y \upharpoonright n = x \upharpoonright n\}.$$

A point $p \in C$ is called *of density* 1 for A if

$$\lim_{n\to\infty} 2^n \mu(p \upharpoonright n, A) = 1. \tag{1}$$

Recall the *density theorem* for C.

THEOREM (Lebesgue). *For every μ-measurable set $A \subseteq C$ we have*

$$\mu\{p \in A\colon p \textit{ satisfies } (1)\} = \mu(A).$$

The game. Given any μ-measurable set $A \subseteq C$ with $\mu(A) > 0$ we have to choose the consecutive coordinates $p_0, p_1, \ldots$ of a point $p \in C$ so as to satisfy (1), however A is not known and every choice p_m has to be decided on account of the values $\mu(x \upharpoonright n, A)$ for a finite set of pairs x, n.

Different classes of strategies for this game may be regarded as admissible (even if not successful).

(a) *Strictly bounded strategies.* p_m is decided only on account of $\mu((p_0, \ldots, p_{m-1}, 0), A)$ and $\mu((p_0, \ldots, p_{m-1}, 1), A)$.

(b) *Bounded strategies.* There exists a function $n\colon \omega \to \omega$ such that p_m is decided on account of the $2^{n(m)}$ values $\mu(x \restriction n(m), A)$.

(c) *Unbounded strategies.* The collection of the values $\mu(x \restriction n, A)$ necessary to decide p_m may depend on those values in this sense that the decision to stop collecting them and computing p_m may depend on the values hitherto collected.

(d) *Random strategies of each of types* (a)-(c). The algorithms for computing p_m may use generators of random numbers.

Now we prove some facts about each kind of strategy.

1. PROPOSITION. *Let S_0 be the strictly bounded strategy which chooses $p_m = 0$ if*

$$\mu((p_0, \ldots, p_{m-1}, 0), A) \geqslant \mu((p_0, \ldots, p_{m-1}, 1), A)$$

and $p_m = 1$ otherwise. Then

$$\lim_{m\to\infty} 2^m \mu(p \restriction m, A) \geqslant \mu(A). \tag{2}$$

The proof is obvious.

Unfortunately, the example which follows shows that neither S_0 nor any other bounded deterministic strategy can secure more than (2).

2. Example. *For every bounded strategy S and $0 < a \leqslant 1$ there exists a Borel set $A \subseteq C$ such that $\mu(A) = a$ but S yields only*

$$\lim_{m\to\infty} 2^m \mu(p \restriction m, A) = a. \tag{3}$$

Proof. Let $n\colon \omega \to \omega$ be the bound for S (see (b)). It is easy to construct a Borel set $A \subseteq C$ such that $\mu(A) = a$ and, if $p \restriction m$ has been chosen by S, then

$$2^{n(m)} \mu(x \restriction n(m), A) = a$$

whenever $x \restriction m = p \restriction m$. Thus A and S yield (3).

3. THEOREM 1. *Let S_1 be the following unbounded deterministic strategy. We choose a sequence $\mu(A) = a_0 < a_1 < \ldots$ with $a_n \to 1$. Then S_1 constructs a sequence of initial segments $p \restriction m_k$ of p in the following way. Let $m_0 = 0$. If $p \restriction m_k$ is already constructed and satisfies*

$$2^{m_k} \mu(p \restriction m_k, A) \geqslant a_k, \tag{4}$$

then S_1 searches for an extension $p \restriction m_{k+1}$ of $p \restriction m_k$ such that

$$2^n \mu(p \restriction n, A) \geqslant a_k \quad \text{for all } n \text{ with } m_k \leqslant n < m_{k+1} \tag{5}$$

and

$$2^{m_{k+1}}\mu(p \upharpoonright m_{k+1}, A) \geqslant a_{k+1}. \tag{6}$$

Then S_1 is always applicable and S_1 secures (1).

Proof. It is obvious that if S_1 is applicable, i.e., the search for the extension $p \upharpoonright m_{k+1}$ satisfying (5) and (6) always succeeds, then S_1 secures (1). Thus it is enough to prove that if $p \upharpoonright m_k$ satisfies (4), then there exists an extension $p \upharpoonright m_{k+1}$ satisfying (5) and (6). Let, for any $x \in C$ and $n < \omega$,

$$V(x \upharpoonright n) = \{y \in C:\ y \upharpoonright n = x \upharpoonright n\}$$

and

$$M_k = V(p \upharpoonright m_k) \cap \{x \in C:\ 2^n\mu(x \upharpoonright n, A) \geqslant a_k \text{ for all } n \geqslant m_k\}.$$

It is enough to prove that

$$\mu(M_k) > 0 \tag{7}$$

since then, by the Lebesgue density theorem, M_k has points of density 1 and long enough initial segments of such points will satisfy (5) and (6). Suppose, to the contrary, that $\mu(M_k) = 0$. Let $N_k = V(p \upharpoonright m_k) - M_k$. Then $\mu(N_k) = 2^{-m_k}$ and for every $x \in N_k$ there exists an $n < \omega$ such that

$$2^n\mu(x \upharpoonright n, A) < a_k \quad \text{and} \quad n > m_k. \tag{8}$$

For any $x \in N_k$ we let $n(x)$ be the least n satisfying (8). Notice that the sets $V(x \upharpoonright n(x))$ with $x \in N_k$ are either equal or disjoint. Hence we have

$$\sum 2^{-n(x)} = 2^{-m_k} \quad \text{and} \quad \mu(p \upharpoonright m_k, A) = \sum \mu(x \upharpoonright n(x), A),$$

where both sums extend over all finite sequences $x \upharpoonright n(x)$ with $x \in N_k$. Then, by (8), we have

$$\mu(p \upharpoonright m_k, A) < \sum 2^{-n(x)}a_k = 2^{-m_k}a_k,$$

which contradicts (4). This contradiction demonstrates (7) and our proof is completed.

The strategy of Theorem 1 has the defect that it may force us to gather an enormous amount of data. It is surprising that a random strategy can save all this trouble.

4. Theorem 2. *Let R be the strictly bounded random strategy which, given $p_0, \ldots, p_{m-1}$, chooses $p_m = 0$ with probability*

$$\pi_m = \frac{\mu((p_0, \ldots, p_{m-1}, 0), A)}{\mu((p_0, \ldots, p_{m-1}), A)} \tag{9}$$

and $p_m = 1$ with probability $1 - \pi_m$. (Notice that R secures its own applicability, i.e., $\mu(p \upharpoonright m, A) > 0$ for every $m < \omega$, with probability 1.) *Then R secures* (1) (*and $p \in A$*) *with probability* 1.

Proof. Let p be a random variable with values in A which is uniformly distributed over A, i.e., for every μ-measurable set $X \subseteq C$ we have

$$\Pr(p \in X) = \frac{\mu(A \cap X)}{\mu(A)}. \tag{10}$$

Notice that for this p, whenever $\mu\big((p_0, \ldots, p_{m-1}), A\big) > 0$,

$$\Pr(p_m = 0 \mid p_0, \ldots, p_{m-1}) = \pi_m$$

as in (9). It follows that the strategy R defines the same uniform probability distribution (10) for p. Thus $\Pr(p \in A) = 1$ and, by the Lebesgue density theorem, (1) holds with probability 1.

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