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A THEOREM IN ASYMPTOTIC NUMBER THEORY

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Let $A(x)$ denote the number of integers of the form pr^2 (p a prime) less than or equal to x. Cohen proved in [1] that

$$A(x) = \frac{\pi^2 x}{6\log x} + O\left(\frac{x}{\log^2 x}\right), \tag{1}$$

and Schwarz [6] sharpened this estimate to

$$A(x) = x\sum_{k=0}^{n} c_k \log^{-k-1} x + O(x\log^{-n-2} x), \tag{2}$$

valid for every natural n with $c_0 = \zeta(2) = \pi^2/6$, $c_k = (-2)^k \zeta^{(k)}(2) + kc_{k-1}$ for $k \geqslant 1$, the O-constant depending on n. The aim of this paper* is to prove a theorem which contains (2) as a special case and to give some applications to the number of integers not exceeding x of a certain form. To do this we shall deal with slowly oscillating functions. A function $L(x)$ is called *slowly oscillating* (or *slowly varying*) if it is positive, continuous for $x \geqslant x_0$ and for every $c > 0$

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1.$$

Karamata [2] gave a canonical representation of these functions in the form

$$L(x) = a\varrho(x)\exp\left[\int_{x_0}^{x} t^{-1}\delta(t)\,dt\right], \tag{3}$$

where $a > 0$, $\varrho(x) \to 1$, $\delta(x) \to 0$ as $x \to \infty$, ϱ and δ being continuous for $x \geqslant x_0$ (for other useful properties see [7]). These functions are of a great number-theoretic significance, since most of the functions like $\log^m x$, $\log\log x$, $\exp[-c\log^b x]$ (for $b < 1$) that appear in the error terms of asymptotic formulas for arithmetical functions are slowly oscillating. Now we are able to formulate the following

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THEOREM. *Let* $a_1 \geqslant 1$ *and* $A = \{a_n\}_{n=1}^{\infty}$ *be an increasing sequence of integers for which*

(a) *for some* $0 < c < 1$,

$$A(x) = \sum_{a \leqslant x, a \in A} 1 = O(x^c).$$

Let $b_1 \geqslant 2$ *and* $B = \{b_n\}_{n=1}^{\infty}$ *be another increasing sequence of integers such that*

(b) *if* $a_i b_j = a_k b_l$ *for some* $i, j, k,$ *and* l, *then* $i = k$ *and* $j = l$;

(c) *for* $C > 0$ *and* $m > 0$,

$$B(x) = \sum_{b \leqslant x, b \in B} 1 = C \int_{b_1}^{x} (\log^{-m} t)\, dt + O\big(xL(x)\big),$$

where $L(x)$ *is a slowly oscillating function that is non-increasing for* x *large enough and satisfies* $L(x) = o(\log^{-m} x)$ *as* $x \to \infty$.

If $N(x)$ *denotes the number of integers not greater than* x *of the form* $n = ab$, $a \in A$, $b \in B$, *then*

$$N(x) = CxF(x) + O\big(xL(x)\big), \tag{4}$$

where

$$F(x) = \int_{a_1}^{y} A(t) t^{-2} (\log x/t)^{-m} dt, \qquad y = [L(x)]^{1/(c-1)},$$

so that $F(x)$ *is a slowly oscillating function asymptotic to* $\log^{-m} x$.

If (a) *and* (b) *hold, but instead of* (c) *we have*

(c′) *for some integer* $n \geqslant 0$,

$$B(x) = C \int_{b_1}^{x} (\log^{-m} t)\, dt + O(x \log^{-m-n-1} x),$$

then

$$N(x) = Cx \sum_{k=0}^{n} \binom{m+k-1}{k} D_k \log^{-m-k} x + O(x \log^{-m-n-1} x), \tag{5}$$

where

$$D_0 = \sum_{n=1}^{\infty} a_n/n, \qquad D_k = (-1)^k F^{(k)}(1) + kD_{k-1} \text{ for } k \geqslant 1,$$

$$F(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

$a_n = 1$ *if* $n \in A$ *and* $a_n = 0$ *otherwise.*

Proof. The decomposition $n = ab$, $a \in A$, $b \in B$ (if it exists) is unique by (b). Thus, if a and b denote elements of A and B, respectively, then

$$N(x) = \sum_{ab \leqslant x} 1 = \sum_{a \leqslant y} \sum_{b \leqslant x/a} 1 + \sum_{b \leqslant z} \sum_{a \leqslant x/b} 1 - \sum_{a \leqslant y} 1 \sum_{b \leqslant z} 1 = S_1 + S_2 - S_3,$$

where $y > 1$ and $yz = x$, and

$$S_3 = A(y)B(z) = O(y^c z \log^{-m} z),$$

$$S_2 = O\Big(x^c \sum_{b \leqslant z} b^{-c}\Big) = x^c \cdot O\Big(B(z)z^{-c} + c\int_{b_1}^{z} B(t)t^{-c-1}dt\Big) = O(y^c z \log^{-m} z),$$

$$S_1 = \sum_{a \leqslant y} B(x/a) = \sum_{a \leqslant y}\Big[C\int_{b_1}^{x/a} (\log^{-m} t)\,dt + O\big(xa^{-1}L(x/a)\big)\Big]$$

$$= C\sum_{a \leqslant y}\int_{b_1}^{x/a} (\log^{-m} t)\,dt + O\Big(x\sum_{a \leqslant y} a^{-1}L(x/a)\Big).$$

If we set

$$H(x, u) = \int_{b_1}^{x/u} C(\log^{-m} t)\,dt$$

and note that $L(z) = L(x/y) \geqslant L(x/a)$ since $L(x)$ is non-increasing, then

$$S_1 = \sum_{a \leqslant y} H(x, a) + O\Big(xL(z)\sum_{a \leqslant y} a^{-1}\Big) = \int_{a_1-0}^{y} H(x, t)\,dA(t) + O\big(xy^{c-1}L(z)\big).$$

Partial integration gives

$$\int_{a_1-0}^{y} H(x, t)\,dA(t) = H(x, y)A(y) - \int_{a_1}^{y} A(t)\,dH(x, t)$$

$$= C\int_{b_1}^{z} (\log^{-m} t)\,dt \cdot O(y^c) + Cx\int_{a_1}^{y} A(t)t^{-2}(\log^{-m} x/t)\,dt$$

$$= CxF(x) + O(y^c z \log^{-m} z),$$

where

$$F(x) = \int_{a_1}^{y} A(t)t^{-2}(\log^{-m} x/t)\,dt.$$

Therefore, we have

$$N(x) = CxF(x) + O(y^c z \log^{-m} z) + O\big(xy^{c-1}L(z)\big). \tag{6}$$

If x is sufficiently large, then $1 < y < x$ for $y = [L(x)]^{1/(c-1)}$ and (4) follows. From (3) it is seen that $y = O(x^\varepsilon)$ for every $\varepsilon > 0$, so that y is small when compared with x, and $CxF(x)$ may be considered as a good main term approximation to $N(x)$. If one chose another value of y, say $y = x^a$ $(0 < a < 1)$, then (6) would give

$$N(x) = CxF(x) + O(x^{1+(c-1)a}\log^{-m} x).$$

But $F(x)$ would then have a much larger interval of integration and the error term would be incorporated in $F(x)$. It is also natural to expect the error term for $N(x)$ not to be smaller than the error term for $B(x)$ which is precisely $O(xL(x))$. $F(x)$ has an asymptotic expansion in terms of negative powers of $\log x$ (which will be shown in proving (6)), but since a finite sum of such powers may give an error term greater than $O(xL(x))$ when $L(x)$ is small, it seems best to leave (4) as it is. $F(x)$ is in any case a slowly oscillating function asymptotic to $\log^{-m} x$. Since $\log^{-m} x$ is slowly oscillating, and a function asymptotic to a slowly oscillating function is slowly oscillating, it is enough to show that

$$\lim_{x\to\infty} \int_{a_1}^{y} A(t)t^{-2}\big(\log x/(\log x/t)\big)^m dt = \int_{a_1}^{\infty} A(t)t^{-2}dt, \tag{7}$$

the last integral existing since $A(t) = O(t^c)$, $0 < c < 1$. Note that y is a slowly oscillating function tending to ∞ as x tends to ∞. Then it follows from (3) by using the de l'Hospital rule that

$$\lim_{x\to\infty} (\log y/\log x) = 0.$$

Thus, for $a_1 \leqslant t \leqslant y$,

$$\lim_{x\to\infty} \big(\log x/(\log x/t)\big)^m = 1 \text{ uniformly in } t,$$

and so for x sufficiently large and for every $\varepsilon > 0$ we have

$$\Big|\int_{a_1}^{y} A(t)t^{-2}\big(\log x/(\log x/t)\big)^m dt - \int_{a_1}^{\infty} A(t)t^{-2}dt\Big|$$

$$\leqslant \int_{a_1}^{y} A(t)t^{-2}\big|\big(\log x/(\log x/t)\big)^m - 1\big|dt + \int_{y}^{\infty} A(t)t^{-2}dt \leqslant \varepsilon \int_{a_1}^{\infty} A(t)t^{-2}dt + O(y^{c-1}),$$

which is arbitrarily small for x sufficiently large.

To obtain (5) from (4) let $L(x) = \log^{-m-n-1} x$ and note that for $0 < q < 1$ it follows from

$$\binom{m+n+k}{k+n+1} \leqslant \binom{n+m}{n+1}\binom{m+k+n}{k}$$

that

$$(1-q)^{-m} = \sum_{k=0}^{n} \binom{m+k-1}{k} q^k + \sum_{k=0}^{\infty} \binom{m+k+n}{k+n+1} q^{k+n+1}$$
$$= \sum_{k=0}^{n} \binom{m+k-1}{k} q^k + O\left(q^{n+1} \sum_{k=0}^{\infty} \binom{m+k+n}{k} q^k\right)$$
$$= \sum_{k=0}^{n} \binom{m+k-1}{k} q^k + O\left(q^{n+1}(1-q)^{-m-n-1}\right).$$

Therefore

$$(8) \qquad \int_{a_1}^{y} A(t)t^{-2}(\log^{-m} x/t)\,dt = \log^{-m} x \int_{a_1}^{y} A(t)t^{-2}(1-\log t/\log x)^{-m}\,dt$$
$$= \sum_{k=0}^{n} \binom{m+k-1}{k} (\log^{-m-k} x) \int_{a_1}^{y} t^{-2} A(t)(\log^k t)\,dt +$$
$$+ O\left(\int_{a_1}^{y} t^{-2} A(t)(\log^{n+1} t)(\log x/t)^{-m-n-1}\,dt\right)$$

and

$$\int_{a_1}^{y} t^{-2} A(t)(\log^k t)\,dt = \int_{a_1}^{\infty} t^{-2} A(t)(\log^k t)\,dt - \int_{y}^{\infty} t^{-2} A(t)(\log^k t)\,dt$$
$$= D_k + O\left(\int_{y}^{\infty} t^{c-2}(\log^k t)\,dt\right) = D_k + O(y^{c-1}\log^k y),$$

where

$$D_k = \int_{a_1}^{\infty} t^{-2} A(t)(\log^k t)\,dt.$$

The O-term in (8) is $O(\log^{-m-n-1} x)$ as in the proof that $F(x)$ is slowly oscillating. Thus

$$N(x) = Cx \sum_{k=0}^{n} D_k \binom{m+k-1}{k} \log^{-m-k} x + O\left(x \sum_{k=0}^{n} (\log^{-m-k} x) y^{c-1} \log^k y\right) +$$
$$+ O(x \log^{-m-n-1} x) = Cx \sum_{k=0}^{n} D_k \binom{m+k-1}{k} \log^{-m-k} x + O(x \log^{-m-n-1} x),$$

since $y^{c-1} = L(x) = \log^{-m-n-1} x$. To evaluate D_k observe that in view of

$$(9) \qquad F(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \int_{a_1-0}^{\infty} t^{-s}\,dA(t)$$

by partial integration we get

$$F(1) = \int_{a_1-0}^{\infty} t^{-1} dA(t) = t^{-1}A(t)\Big|_{a_1-0}^{\infty} + \int_{a_1-0}^{\infty} t^{-2}A(t)dt = \int_{a_1}^{\infty} t^{-2}A(t)dt = D_0,$$

since

$$A(a_1-0) = 0, \quad \lim_{x\to\infty} A(x)/x = 0,$$

and $F(s)$ is absolutely convergent for $\operatorname{Re} s > c$.

Similarly, for $k \geqslant 1$ we have

$$(-1)^k F^{(k)}(s) = \int_{a_1-0}^{\infty} u^{-s}(\log^k u)dA(u),$$

so that for $s = 1$

$$\begin{aligned}(-1)^k F^{(k)}(1) \\ &= \int_{a_1-0}^{\infty} u^{-1}(\log^k u)dA(u) = A(u)u^{-1}\log^k u\Big|_{a_1-0}^{\infty} - \int_{a_1}^{\infty} A(u)d(u^{-1}\log^k u) \\ &= -k\int_{a_1}^{\infty} A(u)u^{-2}(\log^{k-1}u)du + \int_{a_1}^{\infty} u^{-2}A(u)(\log^k u)du = -kD_{k-1} + D_k,\end{aligned}$$

which for $k \geqslant 1$ gives

$$D_k = (-1)^k F^{(k)}(1) + kD_{k-1}. \tag{10}$$

Applications. 1. To see that Schwarz's result (2) follows from (5), let A be the sequence of integer squares, and B the sequence of primes, so that

$$A(x) = [x^{1/2}] = O(x^{1/2}), \quad F(s) = \sum_{n=1}^{\infty} n^{-2s} = \zeta(2s)$$

and, by the prime number theorem (see [8]),

$$B(x) = \pi(x) = \int_2^x (\log^{-1}t)dt + O\big(x\exp[-c\,\delta(x)]\big),$$

where $c > 0$, $\delta(x) = \log^{3/5}x(\log\log x)^{-1/5}$.

Thus (a), (b), and (c) are seen to be satisfied, and (4) gives

$$\sum_{pr^2 \leqslant x} 1 = N(x) = x\int_1^y [t^{1/2}]t^{-2}(\log^{-1}x/t)dt + O\big(x\exp[-c\,\delta(x)]\big), \tag{11}$$

where

$$y = \exp\left[\frac{c}{2}\,\delta(x)\right].$$

Since $\exp[-c\delta(x)] = O(\log^{-N}x)$ for every $N > 0$, (c′) is satisfied with $m = C = 1$, and n arbitrary, Thus (2) follows from (5) and

$$c_0 = F(1) = \zeta(2), \qquad c_k = (-1)^k[\zeta(2s)]_{s=1}^{(k)} + kc_{k-1} = (-2)^k\zeta^{(k)}(2) + kc_{k-1}.$$

2. The previous example can be generalized to the problem of finding the number of integers not greater than x which are a product of a prime p of the form $at+b$ ($(a,b) = 1$) and a number $m = p_1^{a_1}p_2^{a_2}\dots p_i^{a_i}$, where $a_j \equiv c \pmod{d}$ for $j = 1, 2, \dots, i$ and $c \geqslant 0$, $d \geqslant 3$ (if $c = 0$, then m is the d-th power). For A we may take the sequence of m's, and for B the sequence of primes $at+b$, so that $A(x) = O(x^{1/(c+d)})$. Thus (b) is obvious and, by the prime number theorem for primes in arithmetic progressions (see [8]), we have

$$B(x) = [\varphi(a)]^{-1}\int_2^x (\log^{-1}t)\,dt + O\big(x\exp[-c\delta(x)]\big),$$

which implies (c) with $L(x) = \exp[-c\delta(x)]$ or (c′) with $m = 1$, $C = 1/\varphi(a)$, $n \geqslant 0$ arbitrary, and

$$F(s) = \prod_p \big(1+p^{-(c+d)s}+p^{-(c+2d)s}+\dots\big) = \prod_p \frac{1-p^{-ds}+p^{-(c+d)s}}{1-p^{-ds}}.$$

Thus (4) and (5) give the asymptotic formula for $N(x)$.

3. If we assume that the twin prime conjecture or, more generally, the so-called Schinzel H-hypothesis is true (see [4], p. 1-5, and [5]), we may solve problems such as: how many integers not greater than x are products of the d-th power (or an integer m of the type considered in the previous example) and a prime p such that $p' = p+2$ is also a prime ($p'' = p+6$ is also a prime etc.), or a prime p such that p^2+1 is a prime, etc.? In the twin prime case a conjectured formula would be

$$(12)\qquad B(x) = \sum_{p\leqslant x,\, p+2=p'} 1 = 2\prod_{p>2}\big(1-(p-1)^{-2}\big)\int_2^x (\log^{-2}t)\,dt + O(x\log^{-3}x).$$

(a), (b), and (c′) are easily checked so that (5) holds with

$$m = 2, \qquad n = 0, \qquad c = 1/d \leqslant 1/2, \qquad C = 2\prod_{p>2}\big(1-(p-1)^{-2}\big),$$

$$F(s) = \zeta(ds), \qquad D_0 = \zeta(d), \qquad D_k = (-d)^k\zeta^{(k)}(d) + kD_{k-1} \text{ for } k \geqslant 1.$$

4. Finally, let $N(x)$ denote the number of integers not greater than x of the form $n = ab$, $a = p^{2k}$ if $p \not\equiv l \pmod{m}$ (or, more generally, one could take $a = p_1^{a_1}p_2^{a_2}\dots p_i^{a_i}$, $a_t \geqslant 2$ and $p_t \not\equiv l \pmod{m}$ for $t = 1, 2, \dots, i$), $b = q_1^{b_1}q_2^{b_2}\dots q_s^{b_s}$, where $m > 2$, $q_u \equiv l \pmod{m}$ for $u = 1, 2, \dots, s$, and

$q_1, q_2, \ldots, q_s$ primes. If A is the set of all a's (arranged so that elements are increasing) and B the set of b's, then $A(x) = O(x^{1/2})$ and

$$B(x) = Cx\log^{1/\varphi(m)-1}x + O(x\log^{1/\varphi(m)-2}x)$$
$$= C\int_2^x (\log^{1/\varphi(m)-1}t)\,dt + O(x\log^{1/\varphi(m)-2}x),$$

which may be obtained in the same way as the asymptotic formula for integers representable by a sum of two squares was obtained by Postnikov in [3], p. 379-393. Thus (c′) holds with $m \equiv 1/\varphi(m)-1$, C depends on l and m, and (5) may be applied.

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