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*THE THEORY OF TOURNAMENTS: A MINIATURE MATHEMATICAL SYSTEM**

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It is well known that an *axiom system* must have both primitives (undefined terms) and postulates (axioms). There are many practical situations which can be described by the same axiom system. Those include:

1. A round robin tournament in which each player (or team) plays a single game with every other, and exactly one player emerges victorious in any game.

2. The establishment of pecking rights in a flock of hens, where every pair of hens have a go at it and from then on any two hens know which of them does the pecking and which is pecked.

3. A consumer preference relation where say a lady wants to choose just one of five men and makes a choice between every pair of men before reaching a final decision.

4. A task precedence relation wherein several tasks must be accomplished in order to do a complete job and the boss must decide for every pair of tasks which must or should be done first. This situation is called an "assembly schedule" by Foulkes [2].

THE AXIOM SYSTEM

Primitives:

(i) A non-empty set V of p "points", $v_1, v_2, \ldots, v_p$.

(ii) A relation R on the set V, i. e., R is a subset of the cartesian product $V \times V$.

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* This lecture was first given at the Statistics Department of the University of Paris (the Sorbonne) on May 15, 1963. It was most recently given, again in French, on June 15, 1964, at the Department of Mathematics of the Jagellonian University in Kraków, Poland, during its 600 th anniversary year.

Postulates:

P1. R is irreflexive

P2. R is asymmetric

P3. R is complete.

A *tournament* is a model of this axiom system. One can draw a tournament by writing its points as points in the plane and joining a point u to a point v, by a (directed) line uv whenever $(u, v) \in R$. In these terms the meanings of the postulates are as follows:

P1 asserts that there are no "loops", i. e., lines which join a point to itself.

P2 says that there are no symmetric pairs of lines, i. e., not both (u, v) and (v, u) can occur.

P3 stipulates that for any two distinct points u and v, at least one of the ordered pairs (u, v) and (v, u) does occur. Hence by P2 and P3 exactly one of the lines uv and vu occurs in a tournament. In Fig. 1 are shown all tournaments with $p = 1, 2, 3,$ and 4 points. Incidentally the number of tournaments with 5 points is not a power of 2.

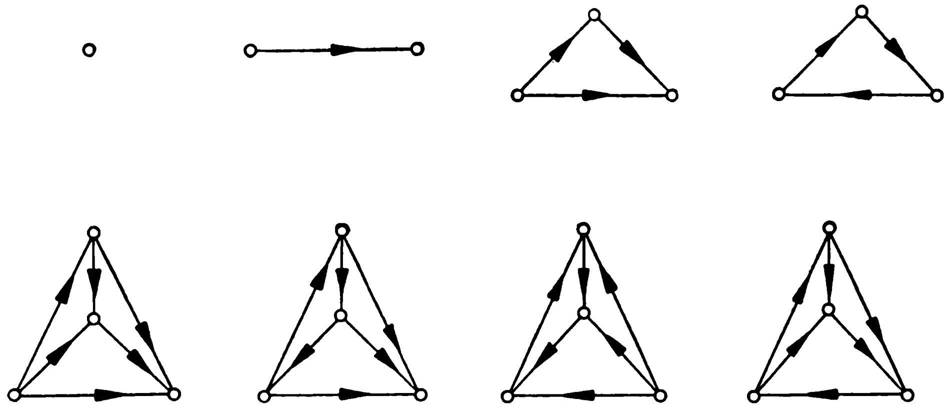

Fig. 1

A *digraph* (directed graph) is a model which satisfies the first postulate. A *complete digraph* satisfies P1 and P3. The *outdegree* of a point v, written od v, is the number of lines from v; the *indegree* id v is the number of lines to v.

It follows directly from the definitions that if v is any point in a tournament T with p points, then $\mathrm{od}\, v + \mathrm{id}\, v = p-1$. Also, the total number of lines in T is $p(p-1)/2$. In the tournament of a round robin competition, the outdegree of a point is the number of victories won by that player. For this reason, we shall call the outdegree of a point v_i, of a tournament its *score*, denoted s_i.

Rather than give precise definitions, we show in Fig. 2 a (directed) path from u to v and a (directed) cycle. The points are distinct in both a path and a cycle.

A *complete path* or a *complete cycle* contains all the points of the tournament. Note that this is not the same usage as the word "complete" in postulate P3, but the meaning will be clear by context. Theorem 1, the first theorem ever found about tournaments, is due to Rédei [8]. It also holds for any complete digraph as observed by König [6]. Its proof is given, but for the later theorems at most a hint of the proof will be indicated. The proof of the other theorems in this article may be found in the review article on tournaments by Harary and Moser [3] or in Chapter 11 of the book on digraphs by Harary, Norman, and Cartwright [4].

Fig. 2

THEOREM 1. *Every tournament has a complete path.*

The proof is given by induction on the number p of points. Referring to Fig. 1, we see that every tournament with 1, 2, 3, or 4 points has a complete path. As the inductive hypothesis, let the theorem hold for all tournaments with n points. Let T by any tournament with $n+1$ points. To complete the proof of the theorem, it is necessary to show that T has a complete path.

Let v_0 be any point of T. Then $T-v_0$ is a tournament with n points. Since the inductive hypothesis applies to $T-v_0$, it has a complete path which may be denoted by $P = v_1v_2v_3\ldots v_n$. Let us return to T and see how the point v_0 can be added to P in order to obtain a complete path of T. Consider the two points v_0 and v_1 of T. By postulates P2 and P3, there are two possibilities: either line v_0v_1 or v_1v_0 is in T. If v_0v_1 is a line of T, then $v_0v_1v_2v_3\ldots v_n$ is a complete path of T. On the other hand, if v_1v_0 is in T, then let v_i be the first point of P, if any, for which the line v_0v_i is in T. Then necessarily line $v_{i-1}v_0$ is in T. Therefore $v_1v_2\ldots v_{i-1}\times$ $\times v_0v_i\ldots v_n$ is a complete path of T (as shown in Fig. 3). But there may not be any such first point v_i, since v_0 might be a receiver of T. In that case, $v_1v_2v_3\ldots v_nv_0$ is a complete path of T, completing the proof.

Rédei also showed that every tournament has an odd number of complete paths but no simple proof is known.

A tournament is called *strongly connected*, or more briefly *strong*, fi for any two points u and v, there is a path from u to v and hence a path from v to u. The next result, proved by Camion [1], gives a criterion analogous to Theorem 1 for a tournament to be strong.

THEOREM 2. *A tournament is strong if and only if it has a complete cycle.*

The following false proof of Theorem 2 actually appeared in the literature. It is left to the reader to find the error in the proof.

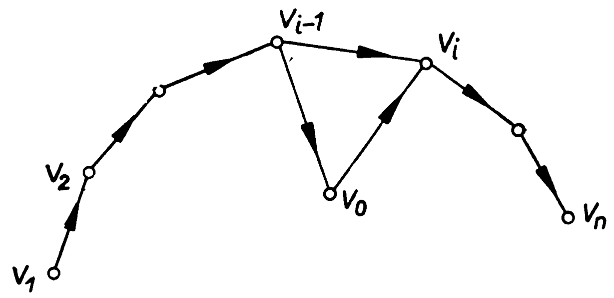

Fig. 3

False proof. Of course every tournament with a complete cycle is strong. To prove the converse by induction, first note that the smallest strong tournament has 3 points and is itself a cyclic triple. Note also from Fig. 1 that there is only one strong tournament with 4 points and it has a complete cycle. Now assume that the theorem is true for every strong tournament with k points and let T be one with $k+1$ points. By the inductive hypothesis, there is a point u in T such that $T-u$ is a strong

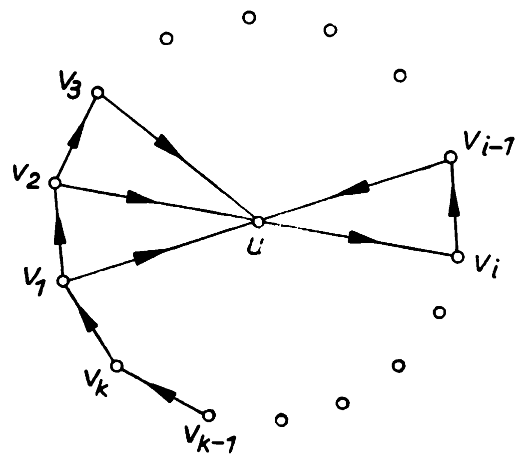

Fig. 4

tournament with k points and hence has a complete cycle $v_1v_2v_3\ldots$ $\ldots v_{k-1}v_kv_1$. As shown in Fig. 4, draw point u inside this cycle and construct a complete cycle of T by the following argument which closely resembles the proof of Theorem 1. Either line vu or line uv is in T by Postulates P2 and P3. Assume vu is in T. Let v_i be the first point such that uv_i is a line of T; then $v_{i-1}u$ is in T. Hence $v_1v_2v_3\ldots v_{i-1}\ uv_i\ldots v_kv_1$ is a complete cycle in T.

We were able to supply a correct proof of Theorem 2 by proving the following more general theorem:

If a tournament is strong, then it contains a cycle of each length $k = 3, 4, \ldots, p$.

The next theorem was actually discovered by biologists who were studying pecking orders in a flock of hens. They found that in practice, the pecking order is not always a transitive tournament, but that cyclic triples may be stable. They also found that in any flock of hens, there is always at least one hen who either pecks every other hen directly, or if not then she pecks a hen who pecks the other hen.

THEOREM 3. *In any tournament, the distance from a point with maximum score to any other point is* 1 *or* 2.

We omit the proof, which uses an argument by contradiction and constitutes an exercise for the reader.

In the theory of relations, see for example Tarski [9], the principle of directional duality asserts that when every concept in a theorem is replaced by its converse concept, the resulting statement is again a theorem. The dual to Theorem 3 so obtained is as follows:

THEOREM 3′. *In any tournament, the distance to a point with minimum score from any other point is* 1 *or* 2.

A tournament is *transitive* if it has no cyclic triples (and hence no cycles at all). In books on the foundations of mathematics, e. g. Wilder [10], a *complete order* (or simple order) is defined axiomatically and coincides with a transitive tournament (except perhaps for the reflexivity postulate, P1). If a judge in a preference relation is consistent, there will be no cyclic triples in his paired comparison choices. Hence the number of cyclic triples in a tournament may be regarded as a measure of inconsistency. For this reason, Kendall and Babington Smith [5] derives the following result which tells the greatest possible number of inconsistent outcomes among triples of choices.

THEOREM 4. *Among all the tournaments with p points, the maximum number of cyclic triples is*

$$c_{\max}(p) = \begin{cases} \dfrac{p^3 - p}{24} & \textit{if } p \textit{ is odd,} \\[2ex] \dfrac{p^3 - 4p}{24} & \textit{if } p \textit{ is even.} \end{cases}$$

There are many proofs that can be given for this result. One of the simplest uses the formula for the number of transitive triples in a given tournament T stated in Theorem 5, subtracts this number from $\binom{p}{3}$, the total number of triples, to obtain the number of cyclic triples, and then applies difference calculus routinely.

THEOREM 5. *The number of transitive triples in a tournament T whose points have scores s_i is* $\sum_{i=1}^{p} s_i(s_i-1)/2$.

The score sequence of a tournament T is the ordered sequence of integers $(s_1, s_2, \ldots, s_p)$. The following theorem by Landau [7] gives a necessary and sufficient condition for a sequence of nonnegative integers to be the scores of some tournament.

THEOREM 6. *A sequence of non-negative integers $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_p$ is a score sequence if and only if their sum satisfies the equation*

$$\sum_{i=1}^{p} s_i = p(p-1)/2, \tag{I}$$

and for every positive integer $k < p$

$$\sum_{i=1}^{k} s_i \geqslant k(k-1)/2. \tag{II}$$

The proof of necessity of (I) and (II) is straightforward. The proof of sufficiency takes a bit longer. There is an analogous result that characterizes the score sequence of a strong tournament. It is obtained from Theorem 6 or replacing the $\geqslant$ inequalities of conditions (II) by strict inequalities.

THEOREM 7. *Let T be a tournament with score sequence $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_p$. Then T is strong if and only if their sum satisfies the equation*

$$\sum_{i=1}^{p} s_i = p(p-1)/2, \tag{I}$$

and for every positive integer $k < p$

$$\sum_{i=1}^{k} s_i > k(k-1)/2. \tag{II}$$

Many other theorems may be derived from these three simple axioms for a tournament, and several difficult unsolved problems remain open.

REFERENCES

[1] P. Camion, *Chemins et circuits hamiltoniens des graphes complèts*, Comptes Rendus 249 (1959), p. 2151-2152.

[2] J. D. Foulkes, *Directed graphs and assembly schedules*, Proceedings of the Symposium on Applied Mathematics, Combinatorial Analysis, American Mathematical Society 10 (1960), p. 281-289.

[3] F. Harary and L. Moser, *The theory of round robin tournaments*, The American Mathematical Monthly 73 (1966), to appear.

[4] F. Harary, R. Norman and D. Cartwright, *Structural models: an introduction to the theory of directed graphs*, New York 1965.

[5] M. G. Kendall and B. Babington Smith, *On the method of paired comparisons*, Biometrika 31 (1940), p. 324-345.

[6] D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig 1936 (reprinted New York 1950).

[7] H. G. Landau, *On dominance relations and the structure of animal societies: III. The condition for a score structure*, The Bulletin of Mathematical Biophysics 15 (1953), p. 143-148.

[8] L. Rédei, *Ein kombinatorischer Satz*, Acta Scientiarum Mathematicarum (Szeged) 1 (1934), p. 39-43.

[9] A. Tarski, *On the calculus of relations*, The Journal of Symbolic Logic 6 (1941), p. 73-89.

[10] R. L. Wilder, *Introduction to the foundations of mathematics*, New York 1952, 2nd edition, 1965.

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