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THE CONVOLUTION EQUATION OF CHOQUET AND DENY FOR PROBABILITY MEASURES ON DISCRETE SEMIGROUPS

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In this note we consider the convolution equation $\mu\lambda = \mu$ for probability measures μ and λ on a discrete semigroup S. It is known that if S is a locally compact topological group, then λ satisfies this equation if and only if $\mu(As^{-1}) = \mu(A)$ for every s in the support of λ and every Borel subset A of S (see Tortrat [11]; the abelian case was considered by Choquet and Deny in [2]; see also [4]). The aim of the present note is to characterize discrete semigroups for which such a theorem remains true. This is done in Corollary 2. We note that some partial results concerning this problem were obtained in [12], [7], [6], and [1]. The study of the convolution equation $\mu\lambda = \mu$ is of some importance in developing the probability theory on semigroups (see [8] and [1]).

For the sake of convenience we shall formulate our results on operands rather than on semigroups.

Let S be a semigroup. By an *operand* X_S over S we mean a set X together with a mapping $(x, s) \to xs$ of $X \times S$ into X satisfying $x(st) = (xs)t$ for all $x \in X$ and $s, t \in S$. In this paper we use the notation and terminology of [3], Chapter 11. Moreover, given an operand X_S over S and subsets $A, B \subseteq X$ and $T \subseteq S$ we put

$$AT = \bigcup \{xt \mid x \in A, t \in T\}, \quad A^{-1}B = \{s \in S \mid A\{s\} \cap B \neq \emptyset\},$$
$$AT^{-1} = \{x \in X \mid A \cap \{x\}T \neq \emptyset\}.$$

If $A = \{a\}$, $B = \{b\}$ or $T = \{t\}$, we shall also write aT, $a^{-1}b$, At^{-1}, etc. By $[T]$ we denote the subsemigroup of S generated by T.

By a *probability measure* μ on X we mean a countably additive non-negative set function defined on 2^X which is regular and the measure of the whole space equals 1 ([5], Section 1). By $C(\mu) = \{x \in X \mid \mu(\{x\}) > 0\}$ we denote the support of μ. The set of all probability measures on X is denoted by $\mathscr{P}_X$. The probability measure concentrated at $x \in X$ is denoted by δ_x.

By $\mathscr{P}_S$ we denote the semigroup of all probability measures on S with the convolution operation (see [5], Section 3). For $\mu \in \mathscr{P}_X$ and $\lambda \in \mathscr{P}_S$ the convolution $\mu\lambda$ of μ and λ is defined by the formula

$$\mu\lambda(A) = (\mu \times \lambda)\{(x, s) \in X \times S \mid xs \in A\}$$
$$= \sum_{s\in S} \mu(As^{-1})\lambda(\{s\}) = \sum_{x\in X} \mu(\{x\})\lambda(x^{-1}A)$$

for an arbitrary $A \subseteq X$.

Let X_S be an operand over S. A subset $N \subseteq X$ is called *invariant* if $NS \subseteq N$. An operand X_S is called *transitive* if $xS = X$ for every $x \in X$. Clearly, X_S is transitive if and only if X contains no proper invariant subset. We say that X_S is *completely reducible* if it is decomposable into transitive suboperands (see [3], p. 257).

LEMMA 1. *Let X_S be an operand over S and let $\mu \in \mathscr{P}_X$ and $\lambda \in \mathscr{P}_S$ satisfy $\mu\lambda = \mu$, $C(\mu) = X$, and $[C(\lambda)] = S$. Then X_S is completely reducible.*

Proof. Let N be an arbitrary invariant subset of X. Then $N \subseteq Ns^{-1}$ for every $s \in S$. Since $\mu\lambda^n = \mu$ (λ^n is the n-fold convolution of λ with itself), we have

$$\sum_{s\in S} \big(\mu(Ns^{-1}) - \mu(N)\big)\lambda^n(\{s\}) = 0$$

for $n = 1, 2, \ldots$ Hence $Ns^{-1} = N$ for every $s \in S$.

Thus we have shown that each invariant subset N of X satisfies $N = Ns^{-1}$ for all $s \in S$. It is easy to see that this property of X_S is equivalent to the assertion of the lemma (cf. [3], Theorem 6.36, (B) and (D)).

LEMMA 2. *Let X_S be a transitive operand over S and let $\mu \in \mathscr{P}_X$ and $\lambda \in \mathscr{P}_S$ satisfy $\mu\lambda = \mu$, $C(\mu) = X$, and $[C(\lambda)] = S$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \lambda^i(x^{-1}A) = \mu(A)$$

for every $A \subseteq X$ and every $x \in X$.

Proof. By our assumptions, $P(x, A) = \lambda(x^{-1}A)$ is a transition probability function with invariant measure μ. Our lemma will follow by the ergodic theorem (see [9], Chapter IV, Corollary 2) if we prove that for every $A \subseteq X$ satisfying $P(x, A) = 1_A(x)$ for all $x \in X$ we have $A = \emptyset$ or X.

Assume that

$$\lambda(x^{-1}A) = \sum_{s\in S} 1_{As^{-1}}(x)\lambda(\{s\}) = 1_A(x)$$

for every $x \in X$. Multiplying this equation by 1_{A^c} we obtain

$$\sum_{s \in S} 1_{A^c}(x) 1_{As^{-1}}(x) \lambda(\{s\}) = 0$$

for every $x \in X$. Hence $1_{A^c} 1_{As^{-1}} = 0$, i.e. $As^{-1} \subseteq A$ for every $s \in C(\lambda)$. Since $[C(\lambda)] = S$ and X_S is transitive, we obtain $A = \emptyset$ or X, which completes the proof.

An operand X_S is called *cancellative* if for all $x, y \in X$ and $s \in S$ such that $xs = ys$ we have $x = y$.

A subset H of a semigroup S is called *left* [*right*] *unitary* if $H^{-1}H \subseteq H$ [$HH^{-1} \subseteq H$], and *unitary* if it is both left and right unitary. A subset H of S is called *strong* if, for every $s, t \in S$, $Hs^{-1} \cap Ht^{-1} \neq \emptyset$ implies $Hs^{-1} = Ht^{-1}$.

Assume that X_S is an operand over S. For each $x \in X$ the set $H_x = \{s \in S \mid xs = x\}$ is called a *stabilizer*. It is known that every stabilizer is a left unitary subsemigroup of S. Moreover, if the operand X_S is cancellative, then every stabilizer is a unitary strong subsemigroup of S (see [10], Section 2, Theorems 2 and 3, Proposition 1).

The main result of this paper is the following

THEOREM. *Let X_S be an operand over a semigroup S and let $\mu \in \mathscr{P}_X$ and $\lambda \in \mathscr{P}_S$ satisfy $\mu\lambda = \mu$, $C(\mu) = X$, and $[C(\lambda)] = S$. Then the following statements are equivalent*:

(i) $\mu\delta_s = \mu$ *for every* $s \in S$;

(ii) X_S *is cancellative*;

(iii) *for every $x \in X$ the stabilizer H_x is a unitary subsemigroup of S.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. Now, assume that $\mu\lambda = \mu$, $C(\mu) = X$, $[C(\lambda)] = S$, and that (iii) holds. From Lemma 1 we infer that X_S is completely reducible. Next, it follows from (iii) and [13], Lemma 1.1, that X_S is cancellative.

Now, without loss of generality we may assume that X_S is transitive and cancellative. Fix an arbitrary $x \in X$ and let

$$\mathscr{R}_{H_x} = \{(s, t) \in S \times S \mid s^{-1}H_x = t^{-1}H_x\}$$

be the Dubreil principal right congruence determined by H_x (see [3], p. 183). By [3], Exercise 1 in Section 11.4, the set H_x is strong and $X_S \cong S/\mathscr{R}_{H_x}$. We prove now that X is finite, i.e. $\operatorname{card}(S/\mathscr{R}_{H_x}) < \infty$.

By [3], Lemma 10.11, the equivalence classes of $\mathscr{R}_{H_x}$ are the non-empty members of $\{H_x s^{-1} \mid s \in S\} \cup \{W_{H_x}\}$, where $W_{H_x} = \{s \in S \mid s^{-1}H_x = \emptyset\}$ is the right residue of H_x. Analogously, the equivalence classes of the Dubreil's principal left congruence

$${}_{H_x}\mathscr{R} = \{(s, t) \in S \times S \mid H_x s^{-1} = H_x t^{-1}\}$$

determined by H_x are the nonempty members of $\{s^{-1}H_x \mid s \in S\} \cup \{{}_{H_x}W\}$, where ${}_{H_x}W = \{s \in S \mid H_x s^{-1} = \emptyset\}$ is the left residue of H_x. Observe that $\operatorname{card}(S/\mathscr{R}_{H_x}) < \infty$ if and only if $\operatorname{card}(S/{}_{H_x}\mathscr{R}) < \infty$. Thus we have to prove that $\operatorname{card}(S/{}_{H_x}\mathscr{R}) < \infty$.

Suppose to the contrary that $\operatorname{card}(S/{}_{H_x}\mathscr{R}) = \infty$ and let k be a natural number such that

$$\mu(\{x\}) > 1/k. \tag{1}$$

There exist $s_1, s_2, \ldots, s_k \in S$ such that the sets $s_1^{-1}H_x, s_2^{-1}H_x, \ldots, s_k^{-1}H_x$ are pairwise disjoint. Since

$$\lambda_n = \frac{1}{n}\sum_{i=1}^{n} \lambda^i$$

is a probability measure on S, we have

$$\sum_{j=1}^{k} \lambda_n(s_j^{-1}H_x) = \lambda_n\Big(\bigcup_{j=1}^{k} s_j^{-1}H_x\Big) \leqslant \lambda_n(S) = 1. \tag{2}$$

On the other hand, applying Lemma 2 we obtain

$$\lim_{n\to\infty} \lambda_n(s_j^{-1}H_x) = \lim_{n\to\infty} \lambda_n\big((xs_j)^{-1}(\{x\})\big) = \mu(\{x\}) \tag{3}$$

for $j = 1, 2, \ldots, k$. Thus (1) and (3) imply

$$\lim_{n\to\infty} \sum_{j=1}^{k} \lambda_n(s_j^{-1}H_x) = k\mu(\{x\}) > 1. \tag{4}$$

Inequalities (2) and (4) give a contradiction.

Thus the set X is finite and the operand X_S is transitive and cancellative. Clearly, the normalized uniform distribution ν on X satisfies $\nu\delta_s = \nu$ for all $s \in S$, and $\nu\lambda = \nu$. Using Lemma 2 we obtain $\nu = \mu$, which gives (i) and completes the proof.

COROLLARY 1. *Let X_S be an operand over a semigroup S. Then the following statements are equivalent:*

(i) *for every $\mu \in \mathscr{P}_X$ and every $\lambda \in \mathscr{P}_S$ such that $\mu\lambda = \mu$ we have $\mu\delta_s = \mu$ for all $s \in C(\lambda)$;*

(ii) *for every $x \in X$ the stabilizer H_x is a unitary subsemigroup of S.*

Proof. If $\mu\lambda = \mu$, then $C(\mu)[C(\lambda)] = C(\mu)$. Thus the implication (ii)$\Rightarrow$(i) follows from the Theorem. Conversely, assume that (ii) does not hold. Thus there exist $x \in X$ and $s, t \in S$ such that $xst = x$, $xt = x$, and $xs \neq x$. Put $y = xs$ and $u = ts$. Then $xt = yt = x$, $xu = yu = y$, $x \neq y$, and $t \neq u$. Set $\mu = (\delta_x + \delta_y)/2$ and $\lambda = (\delta_t + \delta_u)/2$. Then $\mu\lambda = \mu$ and $\mu\delta_t = \delta_x \neq \mu$, i.e. (i) does not hold.

Applying Corollary 1 to the operand S_S we obtain

COROLLARY 2. *Let S be a semigroup. Then the following statements are equivalent:*

(i) *for every $\mu, \lambda \in \mathscr{P}_S$ such that $\mu\lambda = \mu$ we have $\mu\delta_s = \mu$ for all $s \in C(\lambda)$;*

(ii) *for every $a, b, c \in S$ such that $abc = a$ and $ac = a$ we have $ab = a$.*

A further discussion concerning the condition (ii) in Corollaries 1 and 2 can be found in [13]. Observe also that a semigroup S has the property that for every operand X_S over S one of the equivalent conditions in Corollary 1 holds if and only if S is a $\mathscr{T}$-semigroup (see [13]).

The next corollary, which can be easily deduced from the Theorem, shows that condition (ii) in Corollary 1 can be extended to measures.

COROLLARY 3. *Let X_S be an operand over a semigroup S such that for every $x \in X$ the stabilizer H_x is a unitary subsemigroup of S. Then for every $\mu \in \mathscr{P}_X$ the stabilizer $H_\mu = \{\lambda \in \mathscr{P}_S \mid \mu\lambda = \mu\}$ is a unitary (and strong) subsemigroup of $\mathscr{P}_S$.*

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