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REMARKS ON σ-FIELDS WITHOUT CONTINUOUS MEASURES

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Let C be a σ-field on a set X. Denote by $\mathrm{At}\, C$ the family of all atoms of C. We say that C is *separable* if C is countably generated and $\mathrm{At}\, C = \{\{x\}: x \in X\}$. If C is countably generated and $|\mathrm{At}\, C| \geqslant \aleph_1$, then we shall consider the following property of C:

(0) *If μ is a σ-measure on C such that $\mu(X) < \infty$ and $\mu(C) = 0$ for every $C \in \mathrm{At}\, C$, then $\mu(X) = 0$.*

It is easy to see that the question in [1] (P 21) is equivalent to the following one:

(1) Does there exist a set X such that there are countably generated σ-fields C_1 and C_2 on X which do not have property (0) but the σ-field generated by $C_1 \cup C_2$ has property (0)?

In this note we prove that assuming $2^{\aleph_0} = \aleph_1$ or only Martin's Axiom or even only SBCT (Strong Baire Category Theorem) the answer to this question is positive. For definitions of Martin's Axiom see [3]. Let us recall [3] that SBCT is the following sentence:

The intersection of less than $2^{\aleph_0}$ dense open sets on the real line is dense.

For any countably generated σ-fields C_1 and C_2 on X such that $|\mathrm{At}\, C_1|, |\mathrm{At}\, C_2| \geqslant \aleph_0$ we have

$$|\mathrm{At}\, C| = \max\{|\mathrm{At}\, C_1|, |\mathrm{At}\, C_2|\},$$

where C is the σ-field on X generated by $C_1 \cup C_2$. Thus a positive answer to question (1) implies

(2) *There exists a set X on which there are simultaneously a separable σ-field M without property* (0) *and a separable σ-field N with property* (0).

We claim that the converse also holds, i.e., (2) implies the positive answer to question (1).

In fact, let X be a set with the property as in (2). Let

$$X = X_1 \cup X_2, \quad X_1 \cap X_2 = \emptyset \quad \text{and} \quad |X_1| = |X_2|.$$

Since $|X_1| = |X_2| = |X|$, the sets X_1 and X_2 have the property as in (2). Let M_i and N_i be separable σ-fields on X_i such that M_i has not property (0) and N_i has property (0), $i = 1, 2$. Let C_1 be the σ-field on X generated by $M_1 \cup N_2$ and let C_2 be the σ-field on X generated by $N_1 \cup M_2$. Let μ_1 be a σ-measure on M_1 such that $\infty > \mu_1(X_1) > 0$ and $\mu_1(\{x\}) = 0$ for every $x \in X_1$. Let μ_2 be a similar measure on M_2. To see that C_1 and C_2 have not property (0) it is enough to put

$$\nu_1(C) = \mu_1(C \cap X_1) \quad \text{for every } C \in C_1$$

and

$$\nu_2(C) = \mu_2(C \cap X_2) \quad \text{for every } C \in C_2.$$

Now we prove that the σ-field C generated by $C_1 \cup C_2$ has property (0). Suppose not. Then there is a σ-measure μ on C such that $\infty > \mu(X) > 0$ and $\mu(\{x\}) = 0$ for every $x \in X$. Hence, say, $\mu(X_1) > 0$. Define the σ-measure ν on N_1 by

$$\nu(N) = \mu(N \cap X_1) \quad \text{for every } N \in N_1.$$

Thus N_1 has not property (0). A contradiction. In case $\mu(X_2) > 0$ the proof is similar.

So, in order to give an affirmative answer to question (1) it is enough to find a set X with the property as in (2). It is well known that if we assume $2^{\aleph_0} = \aleph_1$, then the real line R has the property as in (2). As a separable σ-field without property (0) we may take the σ-field of Borel sets on the real line R. (Here we do not use $2^{\aleph_0} = \aleph_1$.) It is clear that in order to prove the existence of a separable σ-field on R with property (0) it is enough to find such a σ-field on some set of cardinality $2^{\aleph_0}$. Luzin proves (see [2]) that $2^{\aleph_0} = \aleph_1$ implies that there exists a subset L of R such that $|L| = 2^{\aleph_0}$ and $|L \cap S| < 2^{\aleph_0}$ for every meagre set $S \subset R$. Poprużenko and, more directly, Marczewski prove (assuming $2^{\aleph_0} = \aleph_1$) that the σ-field of Borel sets on L has property (0) (see [2], Section 40, IX, Theorem 1). Martin and Solovay in the proof of the Theorem in [3], p. 175, prove in fact that all what we have said about L assuming $2^{\aleph_0} = \aleph_1$ remains true assuming only SBCT. So also SBCT implies that R has the property as in (2).

Remark 1. In [5] and, in a different way, in [4] it is proved (in ZFC) that on a set X of cardinality $\aleph_1$ there exists a separable σ-field with property (0).

Remark 2. It may be observed (see [3], p. 175) that SBCT implies that every set with the property as in (2) must have cardinality $2^{\aleph_0}$.

Remark 3. It is easy to see that question (1) is equivalent to the following one:

Do there exist countably generated σ-fields C_1 and C_2 on the real line which have not property (0) but the σ-field generated by $C_1 \cup C_2$ has property (0)?

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