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REMARKS ON PRODUCTS OF σ-IDEALS

BY

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1. Recall that an ideal $\mathcal{I}$ of a Boolean algebra $\mathcal{A}$ is called *$\varkappa$-saturated*, where $\varkappa$ is an infinite cardinal, if, whenever $A \subset \mathcal{A} \setminus \mathcal{I}$ has cardinality $\varkappa$, there are distinct $a, b \in A$ such that $a \cap b \notin \mathcal{I}$ (for the notions concerning Boolean algebras, see [19] and [5], pp. 7–16). If $\mathcal{I}$ is a σ-ideal in a Dedekind σ-complete Boolean algebra, it will be ω_1-saturated if and only if there is no uncountable disjoint set $A \subset \mathcal{A} \setminus \mathcal{I}$ (cf. [5], 12 D(d), p. 10). We shall only consider the above case; namely, $\mathcal{A}$ will always be equal to the σ-field $\mathcal{B}(X)$ of Borel subsets of a fixed topological space X. We shall say that a σ-ideal $\mathcal{I} \subset \mathcal{P}(X)$ ($\mathcal{P}(X)$ denotes the power set of X) fulfils the *countable chain condition* (abbr. ccc) if $\mathcal{I} \cap \mathcal{B}(X)$ is ω_1-saturated (in $\mathcal{B}(X)$) which (by the above) means that there is no uncountable family $\mathcal{F} \subset \mathcal{B}(X) \setminus \mathcal{I}$ of disjoint sets. Moreover, we assume that σ-ideals $\mathcal{I} \subset \mathcal{P}(X)$ are proper (i.e., $X \notin \mathcal{I}$) and they contain all singletons $\{x\}$, $x \in X$.

As simple exercises we get

1.1. PROPOSITION. *If $\mathcal{I}$ and $\mathcal{J}$ are σ-ideals in $\mathcal{P}(X)$ such that $\mathcal{I} \subset \mathcal{J}$ and $\mathcal{I}$ fulfils ccc, then $\mathcal{J}$ fulfils it.*

1.2. PROPOSITION. *Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals in $\mathcal{P}(X)$. Then $\mathcal{I} \cap \mathcal{J}$ fulfils ccc if and only if both $\mathcal{I}$ and $\mathcal{J}$ fulfil it.*

For a σ-ideal $\mathcal{I} \subset \mathcal{P}(X)$, put

$$\mathcal{I}^\circ = \{A\colon\ A \subset B \text{ for some } B \in \mathcal{B}(X) \cap \mathcal{I}\}.$$

Then $\mathcal{I}^\circ$ forms a σ-ideal and, by the definition, $\mathcal{I}^\circ$ fulfils ccc if and only if $\mathcal{I}$ fulfils it. We call $\mathcal{I}$ a *Borel σ-ideal* if $\mathcal{I} = \mathcal{I}^\circ$.

The family of meager sets (i.e., those of the first category) and the family of sets having Lebesgue measure zero are well-known Borel σ-ideals fulfilling ccc on the real line. The family of linear σ-porous sets [20] and the σ-ideals defined by Mycielski [15] in the Cantor set are Borel and do not fulfil ccc.

In the paper we are interested in the so-called products of σ-ideals. Now, we shall describe these notions.

Let X and Y be fixed topological spaces. In the sequel, $\mathcal{I}$ and $\mathcal{J}$ will always denote σ-ideals in $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, respectively. We define

$$V(\mathcal{I}, \mathcal{J}) = \{E \subset X \times Y\colon \{x \in X\colon E_x \notin \mathcal{J}\} \in \mathcal{I}\},$$
$$H(\mathcal{I}, \mathcal{J}) = \{E \subset X \times Y\colon \{y \in Y\colon E^y \notin \mathcal{I}\} \in \mathcal{J}\}.$$

These are σ-ideals in $\mathcal{P}(X \times Y)$ called *products* of $\mathcal{I}$ and $\mathcal{J}$ (cf. [1], [7], [14]).

Throughout the paper, $\mathcal{K}$ denotes the family of all meager subsets of the space which is considered at the given moment; if necessary, we use indices, e.g., $\mathcal{K}_X$, $\mathcal{K}_Y$.

The results on $H(\mathcal{I}, \mathcal{J})$ will be omitted since they are analogous to those on $V(\mathcal{I}, \mathcal{J})$.

2. In this section we shall prove and discuss the following theorem:

2.1. THEOREM. *If $V(\mathcal{I}, \mathcal{J})$ fulfils ccc, then both $\mathcal{I}$ and $\mathcal{J}$ fulfil it.*

Proof. Suppose that $\mathcal{I}$ does not fulfil ccc. Then there is a family $\{C_\alpha\colon \alpha < \omega_1\}$ of pairwise disjoint sets from $\mathcal{B}(X) \setminus \mathcal{I}$. Thus, let us observe that the family $\{C_\alpha \times Y\colon \alpha < \omega_1\}$ consists of pairwise disjoint sets from $\mathcal{B}(X \times Y) \setminus V(\mathcal{I}, \mathcal{J})$. Similarly, if $\mathcal{J}$ does not fulfil ccc, we get a family $\{D_\alpha\colon \alpha < \omega_1\}$ of pairwise disjoint sets from $\mathcal{B}(Y) \setminus \mathcal{J}$ and check that the family $\{X \times D_\alpha\colon \alpha < \omega_1\}$ consists of pairwise disjoint sets from $\mathcal{B}(X \times Y) \setminus V(\mathcal{I}, \mathcal{J})$.

Now, our aim is to show that the converse to Theorem 2.1 need not hold; however, the counterexamples given below require some additional set-theoretic axioms. By CH we denote the Continuum Hypothesis, and by MA and SH Martin's Axiom and Suslin's Hypothesis, respectively (cf. [10]).

Recall (cf. [2], p. 35) that a topological space is said to fulfil *ccc* (and called a *ccc space*) if there is no uncountable family of nonempty pairwise disjoint open sets.

It is easy to prove the following propositions:

2.2. PROPOSITION. *A Baire space X fulfils ccc if and only if $\mathcal{K}$ fulfils it.*

2.3. PROPOSITION. *If X, Y are Baire spaces and $X \times Y$ does not fulfil ccc, then $V(\mathcal{K}_X, \mathcal{K}_Y)$ does not fulfil it.*

Now, we quote two results concerning products of ccc spaces.

2.4. THEOREM (R. Laver and F. Galvin; see [6] and [2], Theorem 7.13). *Assume* CH. *There is an extremally disconnected, compact Hausdorff ccc space X such that $X \times X$ is not a ccc space.*

2.5. THEOREM (D. Kurepa; see [18], p. 15). *Assume not-*SH. *There is a compact, connected, hereditarily Lindelöf, first countable, perfectly normal, hereditarily ccc space X such that $X \times X$ is not a ccc space.*

We should add that, under MA + not-CH, any product of ccc spaces is a ccc space (see any of [2], [5], [10], [18]). Note also that MA+not-CH implies SH and that both CH and not-SH are true under Gödel's axiom $V = L$ (cf. [10]).

Since every compact space is a Čech-complete space and this last one is a Baire space ([3], p. 253), Theorems 2.4, 2.5 and Propositions 2.2, 2.3 give the following corollaries yielding the counterexamples announced above.

2.6. COROLLARY. *Assume* CH. *There is an extremally disconnected, compact Hausdorff space X such that $\mathscr{K}$ fulfils ccc and $V(\mathscr{K}, \mathscr{K})$ does not fulfil it.*

2.7. COROLLARY. *Assume not*-SH. *There is a compact, connected, hereditarily Lindelöf, first countable, perfectly normal space X such that $\mathscr{K}$ fulfils ccc and $V(\mathscr{K}, \mathscr{K})$ does not fulfil it.*

3. While one studies ccc for $V(\mathscr{I}, \mathscr{J})$, the following property of $\mathscr{J}$ is helpful (cf. (**) in [7]):

A σ-ideal $\mathscr{J} \subset \mathscr{P}(X)$ is said to be *X-regular* if, for each Borel set $B \subset X \times Y$, the set $\{x \in X: B_x \notin \mathscr{J}\}$ is Borel in X.

Gavalec obtained a criterion ([7], Theorem 2.3) for fulfilling ccc by $V(\mathscr{I}, \mathscr{J})$ (he considers ideals, not necessarily σ-ideals) and required the X-regularity of $\mathscr{J}$ (cf. Theorem 2.2 in [7]). In particular, he showed that $V(\mathscr{K}, \mathscr{L})$ fulfils ccc, where $\mathscr{K}, \mathscr{L} \subset \mathscr{P}([0, 1])$ and $\mathscr{L}$ denotes the family of all Lebesgue null sets. We shall extend this result by considering abstract spaces and measures.

At first, X-regularity will be verified.

A family $\mathscr{F}$ of nonempty open subsets of Y is called a *π-base* (cf., e.g., [5], p. 246) if each nonempty open set includes a member of $\mathscr{F}$.

3.1. PROPOSITION (see [21], Corollary 1.8 (b); cf. also [9], Theorem 2.1.2). *If Y is a Baire space with a countable π-base, then $\mathscr{K} \subset \mathscr{P}(Y)$ is X-regular for every topological space X.*

Further, let μ be a fixed complete measure defined on a σ-field $\mathscr{S} \subset \mathscr{P}(Y)$ containing all Borel subsets of Y and let $\mu(Y) > 0$. Denote by $\mathscr{L}$ the family of all sets having measure μ zero.

We say that μ is *τ-additive* if $\mu(\bigcap F_\alpha) = \inf \mu(F_\alpha)$ for each decreasingly directed family $\{F_\alpha\}$ of closed sets (an equivalent condition can be formulated for open sets; cf. [8] and [4], p. 275).

3.2. PROPOSITION. *Assume that μ is σ-finite.*

(a) *If X is first countable, then $\mathscr{L}$ is X-regular.*

(b) *If μ is τ-additive, then $\mathscr{L}$ is X-regular for every X.*

Proof. It follows from [8] that if X is first countable or μ is τ-additive, then $\mu(G_x)$, as a function of $x \in X$, is lower semicontinuous for each open set G in $X \times Y$, so is Borel measurable. Thus, by using the σ-finiteness of μ, it is not difficult to show that $\mu(B_x)$, as a function of $x \in X$, is Borel measurable for each $B \in \mathscr{B}(X \times Y)$ (we assume first that μ is finite and proceed by induction over the hierarchy of Borel sets in $X \times Y$; cf. the proof of Theorem 2.2 in [7], see also Theorem 2.1.1 in [9]).

Now, from the above propositions and Gavalec's theorem we can derive the following

3.3. COROLLARY. *Let X be a ccc Baire space with a countable π-base. Let μ be a σ-finite complete measure on a σ-field $\mathscr{S} \subset \mathscr{P}(Y)$ containing all Borel subsets of Y, such that $\mu(Y) > 0$. Assume that either μ is τ-additive or Y is first countable. Let*

$$\mathscr{K} \subset \mathscr{P}(X) \quad \text{and} \quad \mathscr{L} = \{A\colon \mu(A) = 0\}.$$

Then $V(\mathscr{K}, \mathscr{L})$ and $V(\mathscr{L}, \mathscr{K})$ fulfil ccc.

Proof. Observe that we may only consider the case where μ is finite. Let

$$\mu(Y) = a \quad \text{and} \quad \mathscr{L}^r = \{A \in \mathscr{B}(Y)\colon \mu(A) \leqslant r\},$$

where r runs over the set Q of all rationals from $[0, a)$. Let $\{U_n\colon n \in N\}$ be a fixed countable π-base of X (N denotes the set of positive integers). Put

$$\mathscr{K}^n = \{A \in \mathscr{B}(X)\colon U_n \cap A \in \mathscr{K}\}, \qquad n \in N.$$

Write $L = \{\mathscr{L}^r\colon r \in Q\}$, $K = \{\mathscr{K}^n\colon n \in N\}$ and, in the same way as in [7], Theorem 1.1, extend K to the system $\bar{K}$. The Gavalec's criterion ([7], Theorem 2.3) applied to $\bar{K}$, L gives the assertion (note that our assumptions taken from Propositions 3.1 and 3.2 are essentially used in the proof; cf. [7], Theorem 2.1).

3.4. EXAMPLE. We shall show that the situation when X has no countable π-base and $V(\mathscr{K}_X, \mathscr{J})$ fulfils ccc is possible. Let X be a ccc Baire space which has no countable π-base, e.g., the Cantor cube $\{0, 1\}^{\omega_1}$ (cf. [3], 3.12.12(a), p. 292). Let Y be a Baire ccc space with a countable π-base. Applying the general version of the Kuratowski–Ulam theorem ((1.1) in [16]), we get

$$V(\mathscr{K}_X, \mathscr{K}_Y) \supset \mathscr{K}_{X \times Y}.$$

Observe that $X \times Y$ is a Baire space ([16], Theorem 2) fulfilling ccc ([5], Corollary 12 J, p. 14); thus, by Proposition 2.2, $\mathscr{K}_{X \times Y}$ fulfils ccc and, by Proposition 1.1, $V(\mathscr{K}_X, \mathscr{K}_Y)$ also fulfils it (in fact, we can prove that $V(\mathscr{K}_X, \mathscr{K}_Y)^\circ = \mathscr{K}_{X \times Y}$ by using the converse to the Kuratowski–Ulam theorem; see Theorem 15.4 in [17]).

Now, assume that $\mathscr{K}, \mathscr{L} \subset \mathscr{P}(X)$, $X = [0, 1]$, where $\mathscr{L}$ is the family of Lebesgue null sets. Recall some observations of Mendez [14]. Let $A \in \mathscr{K}$ and $B \in \mathscr{L}$ be disjoint Borel sets such that $A \cup B = X$ (cf. [17]). Then the sets

$A \times X$ and $B \times X$ belong to $V(\mathcal{K}, \mathcal{L})$ and $V(\mathcal{L}, \mathcal{K})$, respectively, and their union gives $X \times X$. Hence $V(\mathcal{K}, \mathcal{L})$ and $V(\mathcal{L}, \mathcal{K})$ are not included in each other. Let $\mathcal{K}^{(2)}$ and $\mathcal{L}^{(2)}$ denote the σ-ideals in $\mathcal{P}(X \times X)$ of all meager sets and of all two-dimensional Lebesgue null sets, respectively. Then $(A \times A) \cup (B \times B)$ belongs to $V(\mathcal{K}, \mathcal{L}) \cap V(\mathcal{L}, \mathcal{K})$, $(A \times B) \cup (B \times A)$ belongs to $\mathcal{K}^{(2)} \cap \mathcal{L}^{(2)}$ and

$$(*) \qquad ((A \times A) \cup (B \times B)) \cup ((A \times B) \cup (B \times A)) = X \times X.$$

Thus, neither $V(\mathcal{K}, \mathcal{L})$ nor $V(\mathcal{L}, \mathcal{K})$ can be included in any σ-ideals $\mathcal{K}^{(2)}$, $\mathcal{L}^{(2)}$, and all converse inclusions are also impossible. Hence, in this case, ccc for $V(\mathcal{K}, \mathcal{L})$, $V(\mathcal{L}, \mathcal{K})$ cannot be deduced from ccc for $\mathcal{K}^{(2)}$, $\mathcal{L}^{(2)}$ by using Proposition 1.1.

Observe that the set $(A \times A) \cup (B \times B)$ in the decomposition $(*)$ belongs, in fact, to the σ-ideal

$$\mathcal{I}_1 = \{E \subset X \times X \colon E \subset (C \times C) \cup (D \times D) \text{ for some } C \in \mathcal{K}, D \in \mathcal{L}\}.$$

Clearly, $\mathcal{I}_1 \subset \mathcal{I}_2$, where

$$\mathcal{I}_2 = V(\mathcal{K}, \mathcal{L})^\circ \cap V(\mathcal{L}, \mathcal{K})^\circ \cap H(\mathcal{K}, \mathcal{L})^\circ \cap H(\mathcal{L}, \mathcal{K})^\circ$$

(note that, by [14], Theorem 1.3, $V(\mathcal{K}, \mathcal{L})$ and the remaining three products are not Borel σ-ideals). By Gavalec's result and Proposition 1.2, the σ-ideal $\mathcal{I}_2$ fulfils ccc. The following question arises:

3.5. Problem (**P 1356**). Does $\mathcal{I}_1 = \mathcal{I}_2$? If not, does $\mathcal{I}_1$ fulfil ccc?

Finally, observe that it is possible to extend the above considerations to more general settings (cf. [13]).

4. We shall say that a σ-ideal $\mathcal{I} \subset \mathcal{P}(X)$ fulfils the *strong countable chain condition* (abbr. sccc) if, for each family $\{D_\alpha \colon \alpha < \omega_1\}$ of sets from $\mathcal{B}(X) \setminus \mathcal{I}$, there is an uncountable $T \subset \{\alpha \colon \alpha < \omega_1\}$ such that $\bigcap_{\alpha \in T} D_\alpha$ is nonempty. Obviously, if $\mathcal{I}$ fulfils sccc, then it fulfils ccc; note that sccc is an analogue to the possessing of caliber ω_1 defined as in [2], p. 22 (conditions of that kind are discussed in [11] in the aspect of various set-theoretic axioms). Observe that Propositions 1.1 and 1.2 remain true with ccc replaced by sccc.

In this section we study connections between the fulfilment of sccc by $\mathcal{I}$, $\mathcal{J}$ and $V(\mathcal{I}, \mathcal{J})$.

4.1. Theorem. *If $V(\mathcal{I}, \mathcal{J})$ fulfils sccc, then both $\mathcal{I}$ and $\mathcal{J}$ fulfil it.*

The proof is analogous to that of Theorem 2.1.

4.2. Theorem. *If*

(a) *both $\mathcal{I}$ and $\mathcal{J}$ fulfil sccc,*

(b) $\mathscr{J}$ *is* X*-regular,*
then $V(\mathscr{I}, \mathscr{J})$ *fulfils sccc.*

Proof. Let us consider an arbitrary family $\{D_\alpha: \alpha < \omega_1\}$ of sets from $\mathscr{B}(X \times Y) \setminus V(\mathscr{I}, \mathscr{J})$. Let

$$D_\alpha = \{x \in X: (B_\alpha)_x \notin \mathscr{J}\}, \quad \alpha < \omega_1.$$

By (b), the sets D_α are Borel in X. Since $B_\alpha \notin V(\mathscr{I}, \mathscr{J})$, we have $D_\alpha \notin \mathscr{I}$. So, by (a), there is an uncountable $T \subset \{\alpha: \alpha < \omega_1\}$ such that $\bigcap_{\alpha \in T} D_\alpha$ is nonempty. Let

$$x_0 \in \bigcap_{\alpha \in T} D_\alpha.$$

The sets $(B_\alpha)_{x_0}$, $\alpha \in T$, belong to $\mathscr{B}(Y) \setminus \mathscr{J}$; thus, by (a), there is an uncountable $T' \subset T$ such that $\bigcap_{\alpha \in T'} (B_\alpha)_{x_0}$ is nonempty. Hence $\bigcap_{\alpha \in T'} B_\alpha$ is nonempty, which yields the assertion.

In a similar way we get

4.3. THEOREM. *If*
(a) $\mathscr{I}$ *fulfils sccc,*
(b) $\mathscr{J}$ *fulfils ccc,*
(c) $\mathscr{J}$ *is* X*-regular,*
then $V(\mathscr{I}, \mathscr{J})$ *fulfils ccc.*

We shall discuss applications to category and measure.

4.4. PROPOSITION. *Assume* MA + *not*-CH. *If* X *is a Baire space with a countable* π*-base and if* $\{D_\alpha: \alpha < \omega_1\} \subset \mathscr{P}(X)$ *is a family of nonmeager sets with the Baire property, then there is an uncountable* $T \subset \{\alpha: \alpha < \omega_1\}$ *such that* $\bigcap_{\alpha \in T} D_\alpha$ *is nonmeager with the Baire property.*

Proof. Let $\{U_n\}$ denote a fixed countable π-base of X. Since the sets D_α are nonmeager with the Baire property, it follows easily that for each D_α there is U_j such that $U_j \setminus D_\alpha$ is meager. For each $\alpha < \omega_1$ let $k(\alpha)$ be the first index j with the above property. Then

$$\{\alpha: \alpha < \omega_1\} = \bigcup_{n=1}^{\infty} \{\alpha: \alpha < \omega_1, k(\alpha) = n\}.$$

Hence there exists a number n_0 such that the set

$$\{\alpha: \alpha < \omega_1, k(\alpha) = n_0\},$$

denoted by T, is uncountable. It follows from MA + not-CH that a union of fewer than ω_1 sets with the Baire property again has the Baire property (cf. [5], Corollary 22 C, p. 46). Thus we conclude that $\bigcap_{\alpha \in T} D_\alpha$ has the Baire property and $\bigcup_{\alpha\in T}(U_{n_0}\setminus D_\alpha)$ is meager, i.e., $U_{n_0}\setminus\bigcap_{\alpha\in T} D_\alpha$ is meager. Since X is a Baire space, $\bigcap_{\alpha\in T} D_\alpha$ is nonmeager.

4.5. COROLLARY. *Assume* MA + *not*-CH. *If X is a Baire space with a countable π-base, then $\mathscr{K}$ fulfils sccc.*

In the sequel, general assumptions on μ and $\mathscr{L}$ will be the same as those preceding Proposition 3.2. Let L^1 denote the space of equivalence classes of μ-integrable functions on Y. For the definition of a quasi-Radon measure, we refer the reader to [4], Section 72 (cf. also [5], p. 275); note that a quasi-Radon measure is τ-additive.

4.6. PROPOSITION. *Assume* MA + *not*-CH. *If μ is a quasi-Radon measure with separable L^1-space and*

$$\{D_\alpha\colon \alpha<\omega_1\}\subset\mathscr{P}(Y)$$

is a family of measurable sets with positive measure, then there is an uncountable $T\subset\{\alpha\colon \alpha<\omega_1\}$ such that $\bigcap_{\alpha\in T} D_\alpha$ is measurable and

$$\mu\Big(\bigcap_{\alpha\in T} D_\alpha\Big)>0.$$

This follows immediately from Exercise 32 P(h), p. 136, in [5]. A complete, locally finite Borel measure on any analytic space is an important example of a quasi-Radon measure with separable L^1-space (cf. [5], p. 128). Thus, in particular, Lebesgue measure on the real line can be considered here (cf. [10], Exercise 27, p. 89).

4.7. COROLLARY. *Assume* MA + *not*-CH. *If μ is a quasi-Radon measure with separable L^1-space, then $\mathscr{L}$ fulfils sccc.*

By combining Propositions 3.1, 3.2, Corollaries 4.5, 4.7 and Theorem 4.2, we get

4.8. COROLLARY. *Assume* MA + *not*-CH. *Let X be a Baire space with a countable π-base and let $\mathscr{K}\subset\mathscr{P}(X)$. Let μ be a σ-finite quasi-Radon measure on Y such that L^1 is separable and let*

$$\mathscr{L}=\{A\colon \mu(A)=0\}.$$

Then both $V(\mathscr{K},\mathscr{L})$ and $V(\mathscr{L},\mathscr{K})$ fulfil sccc.

In Propositions 4.4 and 4.6, it is enough to assume (in Fremlin's notation) the conditions $\mathfrak{p}>\omega_1$ and $\mathfrak{m}_K>\omega_1$, respectively, which are consequences of MA + not-CH (cf. [5], Corollary 22 C, p. 46; Exercise 32 P(h), p. 136, and comments on pp. 1–7). We do not know interesting applications of Theorem 4.3; we can obviously deduce statements similar to Corollary 3.3, however, we then assume MA + not-CH to get sccc in Theorem 4.3 (a).

Finally, we shall show some negative results on sccc.

4.9. Proposition. *Let X be a T_1-space of cardinality ω_1. There is no σ-ideal in $\mathscr{P}(X)$ which fulfils sccc.*

Proof. Consider any σ-ideal $\mathscr{I} \subset \mathscr{P}(X)$. Let $X = \{x_\alpha : \alpha < \omega_1\}$ and put

$$A_\alpha = \{x_\gamma : \gamma > \alpha\} \quad \text{for } \alpha < \omega_1.$$

Then $A_\alpha \in \mathscr{B}(X) \setminus \mathscr{I}$ for all $\alpha < \omega_1$, and

$$\bigcap_{\alpha \in T} A_\alpha = \emptyset$$

for each uncountable $T \subset \{\alpha : \alpha < \omega_1\}$. Thus $\mathscr{I}$ does not fulfil sccc.

Since each uncountable analytic space has cardinality continuum (cf. [12], p. 387), we obtain

4.10. Corollary. *Assume* CH. *In any uncountable analytic space, there is no σ-ideal fulfilling sccc.*

From Corollaries 4.5, 4.7, 4.10 we deduce

4.11. Corollary. *It is independent of* ZFC *that there exists a σ-ideal fulfilling sccc on the real line.*

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