COLLOQUIUM MATHEMATICUM

VOL. XXXVIII 1978 FASC. 2

ON CONTRACTION MAPPINGS

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Ray [1] has proved the following theorem:

If T_1 and T_2 are two mappings of the metric space X into itself such that

(1) $$\varrho(T_1x, T_2y) \leqslant \lambda\varrho(x, y) \quad \text{for all } x, y \text{ in } X,$$

where $0 \leqslant \lambda < 1$, and if for some x_0 in X the sequence $\{x_n\}$ consisting of the points

$$x_{2n+1} = T_1x_{2n}, \quad x_{2(n+1)} = T_2x_{2n+1}, \quad n = 0, 1, 2, \ldots,$$

has a subsequence $\{x_{n_k}\}$ convergent to a point ξ in X, then T_1 and T_2 have the unique common fixed point ξ.

The result of this theorem is an immediate consequence of the following

THEOREM 1. *If T_1 and T_2 are two mappings of the metric space X into itself such that inequality* (1) *holds, then T_1 and T_2 are identical contraction mappings.*

Proof. We have

$$\varrho(T_1x, T_2x) \leqslant \lambda\varrho(x, x) = 0 \quad \text{for all } x \text{ in } X.$$

Thus $T_1 = T_2$ and the result of the theorem follows.

The following theorem also holds:

THEOREM 2. *Suppose that X is a metric space with no isolated points. If T_1 and T_2 are continuous mappings of X into itself such that*

(2) $$\varrho(T_1x, T_2y) \leqslant \lambda\varrho(x, y) \quad \text{for all distinct } x, y \text{ in } X,$$

where $0 \leqslant \lambda < 1$, then T_1 and T_2 are identical contraction mappings.

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(1) See B. K. Ray, *Contraction mappings and fixed points*, Colloquium Mathematicum 35 (1976), p. 223-234, Theorem 2.

Proof. Let x be an arbitrary point in X. Since x is not isolated, there exist sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \neq y_n$, converging to x. Then

$$\varrho(T_1 x_n, T_2 y_n) \leqslant \lambda \varrho(x_n, y_n).$$

Since T_1 and T_2 are continuous, on letting n tend to infinity we obtain

$$\varrho(T_1 x, T_2 x) \leqslant \lambda \varrho(x, x) = 0,$$

and so we again have $T_1 = T_2$. The result of the theorem now follows.

The theorem does not necessarily hold if the metric space has isolated points. To see this, let $X = \{x, y, z\}$ and let

$$\varrho(x, y) = \varrho(y, z) = 1, \qquad \varrho(z, x) = 2.$$

Define continuous mappings T_1 and T_2 on X by

$$T_1 x = T_1 y = T_1 z = T_2 x = T_2 y = x, \qquad T_2 z = y.$$

Inequality (2) holds with $\lambda = \frac{1}{2}$ but T_1 and T_2 are not identical-

Reçu par la Rédaction le 27. 9. 1976