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POSITIVE-COEFFICIENT ELEMENTS OF HARDY-ORLICZ SPACES

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1. Introduction. In this note we give a characterization of positive-coefficient elements of Hardy-Orlicz spaces by means of the conditions imposed on their coefficients, and investigate Orlicz space behaviour of power series on the unit interval with positive coefficients. In some sense, these results generalize the classical Hardy-Littlewood's theorem (see, e.g., [7], XII (6.6)) on L^p-behaviour of trigonometric series with positive decreasing coefficients and, partially, the recent Askey's result [1] on power series. Lemmas of Section 3 are of the Hardy's inequality type (cf. [2], p. 239) and also have their own independent meaning.

2. Preliminaries; Orlicz and Hardy-Orlicz spaces. A non-decreasing continuous real-valued function Φ defined on the non-negative half-line and vanishing only at the origin will be called an *Orlicz function* ($\mathcal{OF}$). Function $\Phi \in \mathcal{OF}$ is said *to satisfy Δ_2-condition for large u* ($\Phi \in \Delta_2^l$) if there are constants $c > 0$ and $u_0 \geqslant 0$ such that $\Phi(2u) \leqslant c\Phi(u)$, $u \geqslant u_0$. In analogous manner one defines Δ_2-condition for all u (Δ_2^a). If $\Phi \in \Delta_2^a$, then $\Phi(au) \leqslant c^{\log_2^a}\Phi(u)$. A convex Orlicz function Φ satisfying the conditions

$$\lim_{u\to 0}\Phi(u)u^{-1} = 0 \quad \text{and} \quad \lim_{u\to\infty}\Phi(u)u^{-1} = \infty$$

is called a *Young function* ($\mathcal{YF}$). Function Φ belongs to $\mathcal{YF}$ if and only if it admits a representation

$$\Phi(u) = \int_0^u \varphi(t)\,dt,$$

where $\varphi(t)$, $t \geqslant 0$, is positive, continuous on the right, non-decreasing and satisfying $\varphi(0) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$. There is

$$\frac{\Phi(u)}{u} \leqslant \varphi(u) < \frac{\Phi(2u)}{u}$$

(see [4], I. 5).

Let $\psi(s)$ denote the function converse to φ or, more precisely, let $\psi(s) = \sup_{\varphi(t)\leqslant s} t$. Then

$$\Psi(v) = \int_0^v \psi(s)\,ds$$

is also a Young function; it is called a *complementary function to Φ in the sense of Young*. One can show that $u\varphi(u) = \Phi(u)+\Psi(\varphi(u))$ ([4], I (2.7)). Let us yet distinguish a class $\mathscr{M}$ of Orlicz functions Φ which satisfy the following condition of Mulholland (cf. [6]):

($\mathscr{M}$) There exist a convex function Λ, $\lambda > 1$ and $0 < a < 1$, such that the inequality $\Lambda(u) \leqslant \Phi^a(u) \leqslant \lambda\Lambda(u)$ holds for all u.

Condition ($\mathscr{M}$) is in no relation to condition Δ_2. There are functions satisfying condition ($\mathscr{M}$) and not belonging to $\mathscr{YF} \cap \Delta_2$ (e.g., $\Phi(u) = e^u-1$) and conversely, there exists a function $\Phi \in \Delta_2^a \cap \mathscr{YF}$ such that $\Phi \notin \mathscr{M}$ (e.g., $\Phi(u) = (1+u)\ln(1+u)-u$).

Let now $L_\Phi(X, \mu)$, where $\Phi \in \Delta_2^a$, be the Orlicz space, i.e., the set of all complex-valued measurable functions f on a measure space (X, μ) such that the modular $\int_X \Phi(|f(x)|)\,d\mu$ is finite. If $\Phi \in \mathscr{YF} \cap \Delta_2^a$, then L_Φ is a Banach space with the norm

$$\|f\|_{\Phi,X,\mu} = \sup \left|\int_X f(x)g(x)\,d\mu\right|,$$

where the supremum is taken over the set of all functions g satisfying the condition

$$\int_X \Psi(|g(x)|)\,d\mu \leqslant 1.$$

Inequality

$$\|f\|_\Phi \leqslant 1+\int_X \Phi(|f(x)|)\,d\mu$$

and the Hölder inequality

$$\left|\int_X f(x)g(x)\,d\mu\right| \leqslant \|f\|_\Phi\|g\|_\Psi$$

both hold true, whenever $f \in L_\Phi$ and $g \in L_\Psi$ (see [4], § 9). For the theory of Orlicz spaces we refer to the Krasnoselskiĭ and Rutickiĭ's monograph [4].

In this paper the Hardy-Orlicz space H_Φ is meant simply to be a closed subspace of $L_\Phi(\langle 0, 2\pi\rangle, dx)$ spanned over trigonometric polynomials of the form

$$f(t) = \sum_{n=0}^{N} a_n e^{int}.$$

Details concerning an analytic function theory approach to Hardy and Hardy-Orlicz spaces can be found in [3] and [5].

3. Auxiliary inequalities. Let us write

$$F(x) = \int_0^x f(t)\,dt.$$

Mulholland proved in [6] (originally for series) the following generalization of Hardy's inequality:

THEOREM 1 (Mulholland). *Let Φ be an Orlicz function. Then a necessary and sufficient condition for $\Phi(u)$ to satisfy the inequality*

$$\int_0^\infty \Phi\big(F(x)/x\big)\,dx \leqslant K\int_0^\infty \Phi\big(f(x)\big)\,dx$$

for all non-negative functions f is $\Phi \in \mathcal{M}$.

We prove the following

LEMMA 1. *Let $X = R^+$ and $d\mu = x^s dx\,(s \leqslant 0)$. If $\Phi \in \mathcal{M}$, then*

$$\int_X \Phi\big(F(x)/x\big)\,d\mu \leqslant K\int_X \Phi\big(f(x)\big)\,d\mu. \tag{1}$$

Proof. By virtue of the assumption and of the Jensen inequality there exists an a, $0 < a < 1$, such that

$$\Phi^a\big(F(x)/x\big) \leqslant \lambda\Big(\int_0^x \Phi^a\big(f(t)\big)x^{-1}\,dt\Big),$$

whence, using the classical Hardy's inequality (see, e.g., [7], (9.16)), we get

$$\int_X \Phi\big(F(x)/x\big)\,d\mu \leqslant \lambda^{1/a}\int_X\Big[\int_0^x \Phi^a\big(f(t)\big)x^{-1}\,dt\Big]^{1/a} d\mu$$
$$\leqslant \lambda^{1/a}K_1(a)\int_X \Phi\big(f(t)\big)\,d\mu.$$

Letting $\lambda^{1/a}K_1 = K$ the proof is completed.

Mulholland's theorem seems to exhaust entirely the question of possible generalizations of Hardy's inequality. However, one might ask if for some $\Phi \notin \mathcal{M}$ the finiteness of the right-hand side of inequality (1) implies the finiteness of the left-hand of this inequality. In its full generality this problem seems to be still open. Nevertheless, the following lemma extends the class of Orlicz functions for which the affirmative answer is known.

LEMMA 2. *Let $\Phi \in \mathcal{YF} \cap \Delta_2^a$, $X = R^+$, and $d\mu = x^s dx$, where $s < -1$. If $f(x)$ is a non-negative function and $xf(x) \in L_\Phi(X, \mu)$, then $F(x) \in L_\Phi$. Moreover,*

$$\int_X \Phi\big(F(x)\big)\,d\mu \leqslant \max\left[\frac{1}{c-1}, \big(2c\,|s+1|^{-1}\|xf(x)\|_{\Phi,X,\mu}\big)^{\lg_2 c}\right].$$

Proof. Let $0 < a < \beta < \infty$. Integrating by parts and applying inequalities mentioned in the previous section we get

$$\int_a^\beta \Phi\big(F(x)\big)x^s\,dx - \big[(s+1)^{-1}\Phi\big(F(x)\big)x^{s+1}\big]_a^\beta$$

$$= -(s+1)^{-1}\int_a^\beta \varphi\big(F(x)\big)xf(x)x^s\,dx$$

$$\leqslant -(s+1)^{-1}\|\varphi\big(F(x)\big)\|_{\Psi,\langle a,\beta,\rangle,\mu}\|xf(x\|_{\Phi,\langle a,\beta,\rangle,}$$

$$\leqslant -(s+1)^{-1}\Big(\int_a^\beta \Psi\big[\varphi\big(F(x)\big)\big]\,d\mu+1\Big)\|xf(x)\|_{\Phi,\langle a,\beta\rangle,\mu}$$

$$= -(s+1)^{-1}\Big(\int_a^\beta F(x)\varphi\big(F(x)\big)\,d\mu - \int_a^\beta \Phi\big(F(x)\big)\,d\mu+1\Big)\|xf(x)\|_{\Phi,\langle a,\beta\rangle\mu}$$

$$\leqslant -(s+1)^{-1}\Big(\int_a^\beta \Phi\big(2F(x)\big)\,d\mu - \int_a^\beta \Phi\big(F(x)\big)\,d\mu+1\Big)\|xf(x)\|_{\Phi,\langle a,\beta\rangle,\mu}$$

$$\leqslant -(s+1)^{-1}\Big((c-1)\int_a^\beta \Phi\big(F(x)\big)\,d\mu+1\Big)\|xf(x)\|_{\Phi,\langle a,\beta\rangle,\mu}.$$

The Lebesgue integrability on the non-negative half-line of the function $\Phi\big(xf(x)\big)x^s$ implies

$$\int_0^x f(t)\,dt = o\Big(\int_0^x t^{-1}\Phi^{-1}(t^{-(s+1)})\Big)$$

(Φ^{-1} is the function converse to Φ), whence we infer that $\big[\Phi\big(F(x)\big)x^{s+1}\big]_a^\beta$ tends to zero as $a \to 0$ and $\beta \to \infty$. Now passing to the limit with $a \to 0$ and $\beta \to \infty$ in the last inequality, we obtain

$$[1+(s+1)^{-1}(c-1)\|xf(x)\|_{\Phi,X,\mu}]\int_X \Phi\big(F(x)\big)\,d\mu \leqslant -(s+1)^{-1}\|xf(x)\|_{\Phi,X,\mu}.$$

Case I. $\|xf(x)\|_{\Phi,X,\mu} \leqslant 2^{-1}|s+1|(c-1)^{-1}$ (if $\Phi \in \mathcal{YF} \cap \Delta_2$, c is necessarily greater than 2). Then the first factor is greater than or equal to 2^{-1}, and

$$\int_X \Phi\big(F(x)\big)\,d\mu \leqslant (c-1)^{-1}.$$

Case II. $\|xf(x)\|_{\Phi,X,\mu} > 2^{-1}|s+1|(c-1)^{-1}$. Let us write

$$f^*(x) = 2^{-1}|s+1|(c-1)^{-1}\|xf(x)\|_{\Phi,X,\mu}^{-1}f(x).$$

Then $\|xf^*(x)\|_{\Phi,X,\mu} \leqslant 2^{-1}|s+1|(c-1)^{-1}$ and with f^* and

$$F^*(x) = \int_0^x f^*(t)\,dt$$

instead of f and F we are in the conditions of case I. Therefore

$$\int_X \Phi[2^{-1}|s+1|(c-1)^{-1}\|xf(x)\|_{\Phi,X,\mu}^{-1}F(x)]\,d\mu \leqslant (c-1)^{-1}.$$

Finally, by virtue of Δ_2-condition,

$$\int_X \Phi(F(x))\,d\mu \leqslant (c-1)^{-1}c^{\lg_2[2(c-1)|s+1|^{-1}\|xf(x)\|]}$$
$$\leqslant \left(2c|s+1|^{-1}\|xf(x)\|_{\Phi,X,\mu}\right)^{\lg_2 c}.$$

Thus the proof of Lemma 2 is completed.

The author was not able to solve the following

PROBLEM. Does Lemma 2 hold true for $s = 0$? **(P 678)**

The positive solution of this problem could help to weaken assumptions of Theorem 2.

4. The main theorem. In this section we assume $\Phi \in \Delta_2^a \cap \mathcal{M} \cap \mathcal{Y}/\mathcal{F}, d\mu = dx$, and μ as a subscript will be omitted. N stands for the set of all positive integers and ν is the measure on N concentrating the mass n^{-2} at the point $n \in N$, and $A_n = a_0 + a_1 + \ldots + a_n$.

THEOREM 2. *Let*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad 0 \leqslant x < 1.$$

If $a_n \geqslant a_{n+1} \geqslant 0$ $(n = 0, 1, \ldots)$, *then the following four statements are equivalent*:

(i) $f(x) \in L_\Phi\langle 0, 1\rangle$;

(ii) $g(t) \equiv f(e^{it}) \in H_\Phi\langle 0, 2\pi\rangle$;

(iii) $\{na_n\} \in L_\Phi(N, \nu)$;

(iv) $\{A_n\} \in L_\Phi(N, \nu)$.

Proof. We shall prove the following implications: (i) $\Leftrightarrow$ (iv), (iv) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (iv). In fact,

$$\int_0^1 \Phi(f(x))\,dx \geqslant \sum_{n=1}^{\infty} \int_{1-1/(n+1)}^{1-1/(n+2)} \Phi\Big[\sum_{k=0}^{n} a_k x^k\Big]$$
$$\geqslant \sum_{n=1}^{\infty} (n+1)^{-1}(n+2)^{-1}\Phi\Big[\sum_{k=0}^{n} a_k\left(1-(n+1)^{-1}\right)^k\Big].$$

Since $(1-(n+1)^{-1})^k \geqslant \text{const} > 0$ for $k = 0, 1, \ldots, n$ and for all $n \in N$, and $\Phi \in \Delta_2^a$, we have

$$\int_0^1 \Phi\big(f(x)\big)\,dx \geqslant A \sum_{n=1}^{\infty} n^{-2}\Phi(A_n)$$

for some constant $A > 0$. This completes the proof.

(iv) $\Rightarrow$ (i).

$$\begin{aligned}\int_0^1 \Phi\big(f(x)\big)\,dx &= \sum_{n=2}^{\infty} \int_{1-1/(n-1)}^{1-1/n} \Phi\big(f(x)\big)\,dx \\ &\leqslant B' \sum_{n=2}^{\infty} n^{-2}\Phi\Big(\sum_{k=0}^{\infty} a_k(1-n^{-1})^k\Big) \\ &\leqslant B' \sum_{n=2}^{\infty} n^{-2}\Phi\Big(\sum_{k=0}^{\infty}\sum_{j=nk}^{n(k+1)} a_j(1-n^{-1})^j\Big) \\ &\leqslant B'' \sum_{n=2}^{\infty} n^{-2}\Phi\Big(\sum_{k=0}^{\infty} e^{-k}\sum_{j=0}^{n(k+1)} a_j\Big) \\ &\leqslant B'' \sum_{n=2}^{\infty} n^{-2}\Phi\Big(\sum_{k=0}^{\infty} e^{-k}(k+1)A_n\Big) \\ &\leqslant B \sum_{n=1}^{\infty} n^{-2}\Phi(A_n).\end{aligned}$$

(iv) $\Rightarrow$ (ii). We shall prove that $\operatorname{Re} g(t)$ and $\operatorname{Im} g(t)$ are both in $L_\Phi\langle 0, \pi\rangle$. Indeed,

$$|\operatorname{Re} g(t)| \leqslant \sum_{k=1}^{n} a_k + \Big|\sum_{k=n+1}^{\infty} a_k \cos kt\Big| \leqslant A_n + \pi a_n t^{-1}, \qquad 0 < t \leqslant 1.$$

The last inequality holds true by virtue of Theorem XII (66) from paper [7]. Consequently, $|\operatorname{Re} g(t)| \leqslant c' A_n$ for $\pi(n+1)^{-1} \leqslant t \leqslant \pi n^{-1}$ and

$$\begin{aligned}\int_0^{\pi} \Phi\big(|\operatorname{Re} g(t)|\big)\,dt &= \sum_{n=1}^{\infty} \int_{\pi/(n+1)}^{\pi/n} \Phi\big(|\operatorname{Re} g(t)|\big)\,dt \\ &\leqslant c \sum_{n=1}^{\infty} n^{-2}\Phi(A_n).\end{aligned}$$

Almost the same proof remains valid for $\operatorname{Im} g(t)$ and so $g \in H_\Phi$ whenever $\{A_n\} \in L_\Phi(N, \nu)$.

(ii) $\Rightarrow$ (iii). Let us write $r(t) = \operatorname{Re} g(t)$ and $R(t) = \int_0^t r(\tau)\,d\tau$. Since $\Phi \in \mathscr{Y}\mathscr{F}$, r is integrable, the series defining r is its Fourier series ([7], V. § 1), and

$$R(t) = \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nt.$$

Thus we can deduce in a way similar to that of [7], Lemma XII. 6.6, that $R(\pi/n) \geqslant D'a_n$. Consequently,

$$\sum_{n=2}^{\infty} \Phi(na_n)n^{-2} \leqslant D'' \sum_{n=2}^{\infty} \Phi\big(nR(\pi/n)\big)n^{-2}$$

$$\leqslant D'' \sum_{n=2}^{\infty} \Phi\big(n|R|(\pi/n)\big)n^{-2}$$

$$\leqslant D \sum_{n=2}^{\infty} \int_{\pi/n}^{\pi/(n-1)} \Phi[|R|(t)/t]\,dt$$

$$= D \int_0^{\pi} \Phi\big(|R|(t)/t\big)\,dt$$

$$\leqslant D \int_0^{\pi} \Phi\big(|\mathrm{Re}\, g(t)|\big)\,dt$$

$$\leqslant D \int_0^{\pi} \Phi\big(|g(t)|\big)\,dt.$$

The penultimate inequality is motivated by Lemma 1.

(iii) ⇒ (iv). Let $a(x)$ be the function equal to a_n if $n-1 \leqslant x < n$ $(n = 1, 2, \ldots)$ and let

$$A(x) = \int_0^x a(t)\,dt.$$

The assumption $\sum \Phi(na_n)n^{-2} < \infty$ implies that $\Phi\big(a(t)t\big)t^{-2}$ is integrable on the positive half-line, and by virtue of Lemma 2 $(s = -2)$ $\Phi\big(A(t)\big)t^{-2}$ is integrable as well. But this is equivalent to the convergence of the series $\sum \Phi(A_n)n^{-2}$. Hence $\{A_n\} \in L_{\Phi}(N, \nu)$.

Remarks. Surveying the proof of Theorem 2 it is easy to observe that almost all implications proved here (all except (ii) ⇒ (iii)) can be shown under the weaker assumption $\Phi \in \mathcal{YF} \cap \Delta_2^a$. Possibility of a weakening assumptions in the implication (ii) ⇒ (iii), and then in the whole Theorem 2 depends upon the solving the problem P 678 from Section 3.

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