COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 1

ON THE TANGENCY OF SETS IN A METRIC SPACE

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1. The notion of tangency of curves has the property of symmetry in Euclidean, but not in metric spaces. A certain sufficient condition, assuring symmetry of the relation of tangency of simple arcs, has been formulated in [1]. The present note generalizes this result in two ways: first, simple arcs are substituted by more universal sets, and next, the Archimedean hypothesis on arcs is weakened by the substitution of the limit of the ratio of arc longitude to chord longitude equal to 1 by an arbitrary finite number greater than or equal to 1.

2. Let $p \in E$. We put

$$(1) \qquad T_p \stackrel{\mathrm{df}}{=} \{\langle A, B\rangle : A \cup B \subset E \wedge p \in A' \wedge \lim_{\substack{x \to p \\ x \in A}} \frac{\varrho(x; B)}{\varrho(x, p)} = 0\}.$$

In the case of $\langle A, B\rangle \in T_p$ we say that the set A has a *tangency* with the set B in the point p.

It is clear that $p \in A' \Rightarrow \langle A, A\rangle \in T_p$.

THEOREM 1. $(\langle A, B\rangle \in T_p \wedge \langle B, C\rangle \in T_p) \Rightarrow \langle A, C\rangle \in T_p$.

Proof. Supposing that $\langle A, B\rangle, \langle B, C\rangle \in T_p$, choose an arbitrary number $\varepsilon > 0$ and put $\eta = \min\{1, \varepsilon/3\}$. Then there exists $\delta > 0$ such that

$$(2) \qquad \bigwedge_{x \in A} \big(0 < \varrho(x, p) < \delta \Rightarrow \varrho(x; B) < \eta\varrho(x, p)\big)$$

and

$$(3) \qquad \bigwedge_{x \in B} \big(0 < \varrho(x, p) < \delta \Rightarrow \varrho(x; C) < \eta\varrho(x, p)\big).$$

Let us take an arbitrary $x \in A$ such that $0 < \varrho(x, p) < \delta/2$. It follows from (2) that there exists $u \in B$ for which

$$(4) \qquad \varrho(x, u) < \eta\varrho(x, p).$$

Thus $u \neq p$ because $\eta \leqslant 1$. Consequently,

$$(5) \qquad 0 < \varrho(u, p) \leqslant \varrho(u, x) + \varrho(x, p) < (\eta+1)\varrho(x, p) < \delta.$$

Since condition (3) yields $\varrho(u; C) < \eta\varrho(u, p)$, for some $v \in C$ we have

$$\varrho(u, v) < \eta\varrho(u, p). \tag{6}$$

It follows from (4), (5) and (6) that

$$\varrho(x; C) \leqslant \varrho(x, v) \leqslant \varrho(x, u) + \varrho(u, v) < \eta\varrho(x, p) + \eta\varrho(u, p)$$
$$< \eta(\eta+2)\varrho(x, p) \leqslant \varepsilon\varrho(x, p),$$

whence

$$\lim_{\substack{x\to p\\ x\in A}} \frac{\varrho(x; C)}{\varrho(x, p)} = 0.$$

Hence $\langle A, C\rangle \in T_p$, q. e. d.

The relation T_p is then reflexive and transitive in the set

$$\{A: A \subset E \wedge p \in A'\}. \tag{7}$$

3. S. Gołąb and Z. Moszner have proved in [1] that the relation T_p is not symmetric on the set I_p of all simple arcs issuing from the point p. They have considered the problem of searching for the possibly large set $P \subset I_p$ such that

$$\bigwedge_{B} \Big(B \in P \Rightarrow \bigwedge_{A \in I_p} \big(\langle A, B\rangle \in T_p \Rightarrow \langle B, A\rangle \in T_p\big)\Big). \tag{8}$$

Paper [1] contains the proof of the theorem which states that (8) holds for the set A_p of all simple rectifiable arcs B issuing from the point p and satisfying the condition

$$\lim_{\substack{x\to p\\ x\in B}} \frac{l(x, p)}{\varrho(x, p)} = 1, \tag{9}$$

where $l(x, p)$ denotes the length of the subarc of B determined by the points x and p.

Denote by $\tilde{A}_p$ the set of all simple rectifiable arcs B issuing from the point p and satisfying the condition

$$\lim_{\substack{x\to p\\ x\in B}} \frac{l(x, p)}{\varrho(x, p)} < +\infty. \tag{10}$$

It is clear that

$$A_p \subset \tilde{A}_p. \tag{11}$$

For an arbitrary set $C \subset E$ we put

$$[C; p] \stackrel{\mathrm{df}}{=} \{\langle x, y\rangle: x \in E \wedge y \in C \wedge \varrho(x; C) < \varrho(x, p) = \varrho(y, p)\}. \tag{12}$$

Let A_p^* be the set defined as follows:

$$(13)\quad A_p^* \overset{\mathrm{df}}{=} \Big\{C\colon C \subset E \wedge p \in C' \wedge \bigvee_{k>0} \Big(\lim_{\substack{\langle x,y\rangle \to \langle p,p\rangle \\ \langle x,y\rangle \in [C;p]}} \big(\varrho(x,y) - k\varrho(x;C)\big)/\varrho(x,p) = 0\Big)\Big\}.$$

Notice that in the definition (13) appears neither a notion of an arc nor its length.

THEOREM 2. $\bigwedge_{p\in E} (\tilde{A}_p \subset A_p^*)$.

Proof. Let $C \in A_p$. Since $C \in I_p$, then there exists a homeomorphism q mapping the closed segment $0; 1$ onto the set C. Let $q(0) = p$. Consider an arbitrary ordered pair $\langle x, y\rangle \in [C; p]$. From the definition (12) it follows that $x \in E$, $y \in C$ and $\varrho(x; C) < \varrho(x, p) = \varrho(y, p)$.

Put

$$t' = \min\{t\colon 0 \leqslant t \leqslant 1 \wedge \varrho(x, q(t)) = \varrho(x; C)\} \quad \text{and} \quad x' = q(t').$$

From $t' \neq 0$ follows $\varrho(x, p) = \varrho(x; C)$. Then $t' > 0$ and

$$(14)\qquad \varrho(x, x') = \varrho(x; C).$$

Therefore $x' \neq p$, and, consequently, $\varrho(p, x') > 0$, whence we obtain

$$(15)\quad |l(p, x') - l(p, y)| \leqslant \frac{l(p, x')}{\varrho(p, x')} |\varrho(p, x') - \varrho(p, x)| + \left|\frac{l(p, x')}{\varrho(p, x')} - \frac{l(p, y)}{\varrho(p, y)}\right| \varrho(p, y).$$

The definition of the length of an arc implies

$$(16)\quad \varrho(x, y) \leqslant \varrho(x, x') + \varrho(x', y) \leqslant \varrho(x, x') + l(x', y) = \varrho(x, x') + |l(p, x') - l(p, y)|.$$

From (14), (15), (16) and from the inequality

$$|\varrho(p, x') - \varrho(p, x)| < \varrho(x, x')$$

we infer that

$$\varrho(x, y) \leqslant \varrho(x, x') + \frac{l(p, x')}{\varrho(p, x')} \varrho(x, x') + \left|\frac{l(p, x')}{\varrho(p, x')} - \frac{l(p, y)}{\varrho(p, y)}\right| \varrho(p, x)$$
$$= \left(1 + \frac{l(p, x')}{\varrho(p, x')}\right) \varrho(x; C) + \left|\frac{l(p, x')}{\varrho(p, x')} - \frac{l(p, y)}{\varrho(p, y)}\right| \varrho(p, x).$$

As the triangle axiom and equality (14) imply

$$\varrho(p, x') \leqslant \varrho(p, x) + \varrho(x, x') = \varrho(p, x) + \varrho(x; C) < 2\varrho(p, x),$$

then, in view of (10), there exists a number $k > 1$ such that $l(p, x')/\varrho(p, x') \leqslant k-1$ and

$$\lim_{\substack{\langle x,y\rangle \to \langle p,p\rangle \\ \langle x,y\rangle \in [C;p]}} \left| \frac{l(p, x')}{\varrho(p, x')} - \frac{l(p, y)}{\varrho(p, y)} \right| = 0.$$

Hence Theorem 2 is proved.

We shall denote by C_p the set of all sets $A \subset E$ such that $p \in \bar{A}$ and that the component of the set $\bar{A}$ containing the point p does not reduce to the point p alone.

THEOREM 3. $\bigwedge_{p \in E} \bigwedge_{B} \big(B \in A_p^* \Rightarrow \bigwedge_{A \in C_p} (\langle A, B\rangle \in T_p \Rightarrow \langle B, A\rangle \in T_p)\big)$.

Proof. Suppose that $B \in A_p^*$ and consider an arbitrary set such that $\langle A, B\rangle \in T_p$. If follows from (13) that there exists a number $k > 0$ such that

$$\lim_{\substack{\langle x,y\rangle \to \langle p,p\rangle \\ \langle x,y\rangle \in [B;p]}} \big(\varrho(x, y) - k\varrho(x; B)\big)/\varrho(x, p) = 0. \tag{17}$$

Consider an arbitrary $\varepsilon > 0$ and put $\eta = \min\{1, (1+k)\varepsilon\}$. By the definition (1) of the set T_p we have

$$\lim_{\substack{x \to p \\ x \in \bar{A}}} \varrho(x; B)/\varrho(x, p) = 0. \tag{18}$$

Let S be the component of the set $\bar{A}$ that contains the point p. Then the set S contains also another point $q \neq p$. Conditions (17) and (18) imply that there exists a number δ such that

$$0 < \delta < \varrho(p, q), \tag{19}$$

$$\bigwedge_{u \in \bar{A}} \big(0 < \varrho(u, p) < \delta \Rightarrow \varrho(u; B) < \eta\varrho(u, p)\big), \tag{20}$$

and

$$\bigwedge_{u,v} \big((\langle u, v\rangle \in [B; p] \wedge 0 < \varrho(u, p) < \delta \wedge 0 < \varrho(v, p) < \delta) \Rightarrow \Rightarrow \varrho(u, v) - k\varrho(u; B) < \eta\varrho(u, p)\big). \tag{21}$$

Consider an arbitrary $x \in B$ such that $0 < \varrho(x, p) < \delta$. In view of the connexivity of the set S and inequality (19) there exists $y \in S$ such that $\varrho(x, p) = \varrho(y, p)$. Since $S \subset \bar{A}$, we have $y \in \bar{A}$. It follows by means of (20) that $\varrho(y; B) < \varrho(y, p) = \varrho(x, p)$. Therefore $\langle y, x\rangle \in [B; p]$. It follows from condition (21) that

$$\varrho(x; A) = \varrho(x; \bar{A}) \leqslant \varrho(x, y) < k\varrho(y; B) + \eta\varrho(y, p).$$

Hence

$$\frac{\varrho(x;A)}{\varrho(x,p)} < k\frac{\varrho(y;B)}{\varrho(y,p)} + \eta < k\eta + \eta \leqslant \varepsilon,$$

which yields

$$\lim_{\substack{x \to p \\ x \in B}} \frac{\varrho(x;A)}{\varrho(x,p)} = 0.$$

Thus $\langle B, A\rangle \epsilon T_p$, q. e. d.

REFERENCES

[1] S. Gołąb and Z. Moszner, *Sur le contact des courbes dans les espaces métriques généraux*, Colloquium Mathematicum 10 (1963), p. 305-311.

[2] C. Kuratowski, *Topologie*, Warszawa 1958.

Reçu par la Rédaction le 28. 12. 1964