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MULTIPLICATIVE OPERATORS ON SYMMETRIC COMMUTATIVE ALGEBRAS

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1. Introduction. Let A be a *-normed commutative Banach algebra (i.e. a commutative Banach *-algebra with isometric involution) with identity and let B be a commutative B^*-algebra. In [3] Espelie proved that the extreme points of the set $P(A, B)$ of all positive operators $T\colon A \to B$ satisfying $\|T\| \leqslant 1$ coincide with the set of multiplicative operators in $P(A, B)$. It was a generalization of a previous result due to A. and C. Ionescu Tulceas [4] who considered algebras $C(X)$ of all continuous real functions on compact Hausdorff spaces, and positive operators T satisfying $T1 = 1$. Some other extensions of the Ionescu Tulceas result were done for algebras of functions (see, e.g., [1]).

In [2] Ellis gave two more equivalent conditions for a positive operator $T\colon C(X) \to C(Y)$ satisfying $T1 = 1$ to be multiplicative. The condition that we are interested in (after a slight modification of its original form in [2]) reads: $T|x| = |Tx|$ for all x in $C(X)$. The spaces $C(X)$ and $C(Y)$ are again assumed to be real.

Our aim is to obtain a more abstract version of the Ellis result. If A is a *-normed symmetric commutative Banach algebra with identity, then we define $|x|$ for any x in a reasonable subset A_0 of A (Section 2). Now, assume that T is a positive operator from A into B, a commutative B^*-algebra with identity, and let $T1 = 1$. Then T is multiplicative if and only if $T|x| = |Tx|$ for all x in A_0 (Corollary 1 in Section 4).

2. Definitions and basic facts. We adopt the terminology of Banach algebras from [5].

Recall that a *-normed commutative Banach algebra A is called *symmetric* if $\varphi(x^*) = \overline{\varphi(x)}$ for any $x \in A$ and for any φ in Φ_A, the set of all non-zero multiplicative functionals on A. It is called a *B^*-algebra* if $\|x^*x\| = \|x\|^2$ for any $x \in A$. Closed *-subalgebras of symmetric algebras (B^*-algebras) are also symmetric algebras (B^*-algebras).

Let A be a *-normed symmetric commutative complex Banach algebra with identity. If a functional f on A is *positive* (i.e. $f(x^*x) \geqslant 0$ for all $x \in A$), then f is *hermitian* (i.e. $f(x^*) = \overline{f(x)}$), bounded with $\|f\| = f(1)$, and satisfies $|f(x)|^2 \leqslant \|f\| f(x^*x)$. Elements of Φ_A are positive functionals and $\Phi_A \cup \{0\}$ coincides with $\mathrm{ext} P_A$, the extreme points of the set P_A of all positive functionals f satisfying $|f(x)|^2 \leqslant f(x^*x)$ (or, equivalently, $\|f\| \leqslant 1$). The *spectrum* $\mathrm{Sp}\,x$ of an element x in A is the set $\{\varphi(x) \colon \varphi \in \Phi_A\}$. If $\mathrm{Sp}\,x \geqslant 0$ for a hermitian element x in A, then $f(x) \geqslant 0$ for any positive functional f on A. If a hermitian element x in A satisfies $0 < \mathrm{Sp}\,x \leqslant 1$, then the series

$$1 + \sum_{n=1}^{\infty} \binom{1/2}{n} (x-1)^n$$

converges in A to a hermitian element $x^{1/2}$ such that $(x^{1/2})^2 = x$. Moreover, $0 < \mathrm{Sp}\,x^{1/2} \leqslant 1$.

Proofs of the above-listed facts can be found in [5].

We denote by A_0 the set of all regular elements x in A satisfying $\|x\| \leqslant 1$. Clearly, the interior of A_0 is not empty. If $x \in A_0$, then $\|x^*x\| \leqslant 1$ so that $0 < \mathrm{Sp}\,x^*x \leqslant 1$ and $(x^*x)^{1/2}$ exists. We define the absolute value of an element x in A_0 by

$$|x| = (x^*x)^{1/2}$$

(this definition coincides with the usual definition of the absolute value if A is a B^*-algebra). By the definition of $|x|$ we obtain $\varphi(|x|) = |\varphi(x)|$ for any x in A_0 and φ in Φ_A.

Let now B be a commutative B^*-algebra and T a linear operator from A into B. We call T *positive* if Tx^*x is of the form y^*y ($y \in B$) for any $x \in A$. Since every hermitian element of A is a difference of two elements of the form x^*x, any positive operator T is *hermitian*, i.e. it satisfies $Tx^* = (Tx)^*$ for any x in A. Moreover, any positive operator T is bounded:

$$\|Tx\| = \sup\{|\varphi Tx| \colon \varphi \in \Phi_B\} \leqslant \|x\| \varphi T1 \leqslant \|x\| \, \|T1\|,$$

where φT is viewed as a positive functional on A. We denote by $P(A, B)$ the set of all positive operators T from A into B satisfying $\|T\| \leqslant 1$. It should be noted that any multiplicative operator is an element of $P(A, B)$. Indeed, if T is multiplicative, then $\varphi T \in \Phi_A \cup \{0\}$ whenever $\varphi \in \Phi_B$ so that $\varphi Tx^*x \geqslant 0$ for any $\varphi \in \Phi_B$, hence Tx^*x is of the form y^*y (B can be treated as the algebra of continuous functions on a compact space). Moreover,

$$\|T\| = \sup\{\|Tx\| \colon \|x\| = 1\} = \sup\{|\varphi Tx| \colon \|x\| = 1, \varphi \in \Phi_B\} \leqslant 1,$$

since $\|\varphi T\| = \varphi T1 = 0$ or 1.

Let us note that if A is semi-simple, then $|x| = x$ for any hermitian element $x \in A_0$ with positive spectrum. In fact, we obtain

$$\big(\varphi(x)\big)^2 = \varphi(x^2) = \varphi(|x|^2) = \big(\varphi(|x|)\big)^2,$$

so that

$$\varphi(|x|-x) = 0 \quad \text{for any } \varphi \text{ in } \Phi_A.$$

3. Functionals preserving the absolute value. Throughout this section, A denotes a *-normed symmetric commutative Banach algebra with identity.

LEMMA 1. *If f is a positive functional on A, then*

$$|f(x)| \leqslant f(|x|) \quad \textit{for any } x \textit{ in } A_0.$$

The lemma follows immediately from the integral representation of positive functionals on A, or by a direct calculation.

LEMMA 2. *If a positive functional f on A with $\|f\| = 1$ satisfies $f(|x|) = |f(x)|$ for any x in A_0, then $f \in \mathrm{ext} P_A$.*

Proof. Suppose that $f = (f_1+f_2)/2$ with $f_j \in P_A$ $(j = 1, 2)$. Then

$$1 = \|f\| = f(1) = \frac{f_1(1)+f_2(1)}{2},$$

implying $f_j(1) = 1$ for $j = 1, 2$. If $x \in A_0$, then, by Lemma 1,

$$|f(x)| \leqslant \frac{|f_1(x)|+|f_2(x)|}{2} \leqslant \frac{f_1(|x|)+f_2(|x|)}{2} = f(|x|) = |f(x)|,$$

whence

$$|f_1(x)+f_2(x)| = |f_1(x)|+|f_2(x)|,$$

so that $\mathrm{Arg} f_1(x) = \mathrm{Arg} f_2(x)$ whenever $f_1(x) \neq 0 \neq f_2(x)$. Suppose that $f_1(x) \neq f_2(x)$ for some x in the interior of A_0. Taking a positive real number δ such that $x+\delta \in A_0$ we obtain

$$\mathrm{Arg}\big(f_1(x)+\delta\big) = \mathrm{Arg}\big(f_2(x)+\delta\big),$$

which, along with the previous equality, implies $\mathrm{Im} f_j(x) = 0$ for $j = 1, 2$. This is easily seen to be a contradiction by putting $x = i/2$.

For semi-simple algebras we have:

LEMMA 3. *If A is semi-simple, then any functional on A satisfying $f(|x|) \geqslant 0$ for all $x \in A_0$ is positive.*

Proof. Let $\|x^*x\| < 1$. For any positive number δ the element $x^*x+\delta$ is regular and lies in A_0 whenever δ is sufficiently small. By semi-simplicity of A we have $|x^*x+\delta| = x^*x+\delta$ (Section 2) so that $f(x^*x+\delta) \geqslant 0$. Therefore, $f(x^*x) \geqslant -\delta f(1)$, since $f(1) = f(|1|) \geqslant 0$. This ensures the positivity of f.

By the proved lemmas and by the last lines of Section 2 we obtain the following result:

THEOREM 1. *Let A be a *-normed symmetric commutative Banach algebra with identity. For any positive functional f on A with $\|f\| = 1$ the following conditions are equivalent*:

(i) $f(|x|) = |f(x)|$ *for any* x *in* A_0,

(ii) f *is multiplicative*,

(iii) $f \in \operatorname{ext} P_A$.

Moreover, if A is semi-simple, then the assumption that f is positive can be neglected.

4. Operators preserving the absolute value. We consider two algebras A and B, the latter being a B^*-algebra. The absolute value in B can be understood in the usual sense and is defined for all elements.

THEOREM 2. *Suppose that A and B are *-normed commutative Banach algebras, A is symmetric and has the identity, and B is a B^*-algebra. If T is a positive linear operator from A into B, then the following two conditions are equivalent*:

(i) $T|x| = |Tx|$ *for any* x *in* A_0,

(ii) $T1Txy = TxTy$ *for any* x *and* y *in* A.

Proof. (i) $\Rightarrow$ (ii). We have $\varphi T|x| = |\varphi Tx|$ for any φ in Φ_B. The functional φT is positive, hence, by Theorem 1, $\varphi T / \varphi T1 \in \Phi_A$ whenever $\varphi T1 \neq 0$. Thus

$$\varphi(T1Txy) = (\varphi Tx)(\varphi Ty) \quad \text{for any } \varphi \text{ in } \Phi_B,$$

so that $T1Txy = TxTy$.

(ii) $\Rightarrow$ (i). For any $x \in A_0$ we have

$$(T|x|)^2 = T1T|x|^2 = T1Tx^*x = Tx^*Tx = |Tx|^2,$$

which implies $T|x| = |Tx|$ (note that $\operatorname{Sp}|x| \geqslant 0$, so that $\varphi T|x| \geqslant 0$ for any φ in Φ_B).

By Theorem 2 of [3] all elements of $\operatorname{ext} P(A, B)$ satisfy the equivalent conditions of our theorem. Conversely, if $T1$ is an idempotent in B (then it is an extreme point of the set $\{x \in B\colon \|x\| \leqslant 1 \text{ and } x \text{ is of the form } y^*y \text{ for some } y \text{ in } B\}$), then $T \in \operatorname{ext} P(A, B)$ ([3], Theorem 5) and T is multiplicative ([3], Theorem 3). Moreover, if A is semi-simple, then every operator T satisfying condition (i) of Theorem 2 is positive, since the functionals φT ($\varphi \in \Phi_B$) satisfy the assumptions of Lemma 3. By these remarks we obtain

COROLLARY 1. *Under the assumptions of Theorem 2 the following conditions are equivalent*:

(i) $T|x| = |Tx|$ *for any* x *in* A_0, *and* $T1$ *is an idempotent*,

(ii) T *is multiplicative*,

(iii) $T \in \operatorname{ext} P(A, B)$.

Moreover, if A is semi-simple, then we do not need to assume that T is positive.

If A and B are commutative B^*-algebras with identities and if $T1 = 1$ holds, then, by an argument similar to that used in Theorem 2, the multiplicativity of T implies $T|x| = |Tx|$ for any x in A. Therefore, Corollary 1 provides the complex version of Ellis' result [2].

It should be noted that in [2] Ellis considered the set

$$P_1(A, B) = \{T: T \text{ is positive and } T1 = 1\}$$

instead of $P(A, B)$. It is, however, clear that

$$P_1(A, B) \subset P(A, B) \quad \text{and} \quad \mathrm{ext} P_1(A, B) = P_1(A, B) \cap \mathrm{ext} P(A, B)$$

if A and B are B^*-algebras.

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