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COMPOSING T-DESIGNS

BY

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1. Preliminaries. Throughout, v, b, r, k, λ and t denote positive integers. The standard binomial coefficient, "v take k", is denoted by $\binom{v}{k}$. For a set X, $|X|$ denotes the cardinality of X. Let $\mathscr{P}$ be a finite set (of points) and $\mathscr{B}$ a list of subsets (called *blocks*) of $\mathscr{P}$. For $X \subseteq \mathscr{P}$, we let $[X]$ denote $\{B \mid B \in \mathscr{B}$ and $X \subseteq B\}$. For a single point $P \in \mathscr{P}$, we write $[P]$ for $[\{P\}]$. For $B \in \mathscr{B}$, we put $B' = \mathscr{P} - B$ and $\mathscr{B}' = \{B' \mid B \in \mathscr{B}\}$.

Definition 1.1. $(\mathscr{P}, \mathscr{B})$ is a (v, b, r, k, λ) *t-design* if

1. $|\mathscr{P}| = v$, $|\mathscr{B}| = b$;
2. for each $P \in \mathscr{P}$, $|[P]| = r$;
3. for each $B \in \mathscr{B}$, $|B| = k$;
4. for each $X \subseteq \mathscr{P}$ with $|X| = t$, $|[X]| = \lambda$.

For $t = 2$, 2-designs are known as block designs. We use these terms interchangeably. To avoid trivial t-designs we assume $v > k > t$. Generally, we also wish to exclude the trivial complete designs which arise when all $\binom{v}{k}$ subsets of $\mathscr{P}$ of the appropriate size are taken as blocks. It should be noted that no nontrivial t-designs are known for $t \geqslant 6$. In the sequel we construct several new families of 2-, 3-, 4-, and 5-designs.

For t-designs the following well-known "coming down" relation holds (see [6] for a proof):

THEOREM 1.1. *If $(\mathscr{P}, \mathscr{B})$ is a (v, b, r, k, λ) t-design, then $(\mathscr{P}, \mathscr{B})$ is also a (v, b, r, k, λ') s-design for any $s < t$, where λ' satisfies*

$$\lambda' \binom{k-s}{t-s} = \lambda \binom{v-s}{t-s}.$$

As notational convenience we will use $\lambda_t, \lambda_{t-1}, \ldots, \lambda_1, \lambda_0$ to denote the sequence of λ's occurring by way of Theorem 1.1 from a (v, b, r, k, λ_t) t-design. The number of blocks through any point is given by λ_1, and thus $\lambda_1 = r$. Likewise, λ_0 is the total number of blocks, and hence $\lambda_0 = b$.

The simplest extension theorem for t-designs is obtained as follows:

If $(\mathscr{P}, \mathscr{B})$ is a (v, b, r, k, λ) t-design, then by taking n distinct copies of the design, all on the point set $\mathscr{P}$, we obtain a $(v, nb, nr, k, n\lambda)$ t-design. Such multiple designs are occasionally useful.

2. Composing t-designs. The theorems of this section combine or enlarge given t-designs to produce new t-designs. The next theorem is a good example of such a composition theorem.

THEOREM 2.1. *If there exist (v, b, r, k, λ) and $(v, b', r', k+1, \lambda')$ t-designs and $b = r+r'$, then there exists a $(v+1, b+b', b, k+1, \lambda+\lambda')$ t-design.*

Proof. Let $(\mathscr{P}, \mathscr{B})$ denote a (v, b, r, k, λ) t-design and $(\mathscr{P}, \mathscr{B}_0)$ a $(v, b', r', k+1, \lambda')$ t-design. Let $\mathscr{P}^+ = \mathscr{P} \cup \{\infty\}$ and $\mathscr{B}^+ = \{B \cup \{\infty\} \mid B \in \mathscr{B}\}$. It remains to show that $(\mathscr{P}^+, \mathscr{B}^+ \cup \mathscr{B}_0)$ is the desired t-design. Obviously, $|\mathscr{P}^+| = v+1$ and $|\mathscr{B}^+ \cup \mathscr{B}_0| = b+b'$. Also ∞, clearly, belongs to b blocks and any point in $\mathscr{P}$ belongs to $r+r' = b$ blocks. Clearly, each block of $\mathscr{B}^+ \cup \mathscr{B}_0$ consists of $k+1$ points. Also any t points of $\mathscr{P}^+$, distinct from ∞, are contained in λ blocks of $\mathscr{B}^+$ and in λ' blocks of $\mathscr{B}_0$, and thus in $\lambda+\lambda'$ blocks in all. Any t points of $\mathscr{P}^+$ including ∞ belong to no blocks of $\mathscr{B}_0$ and to λ_{t-1} blocks of $\mathscr{B}^+$. Hence the proof will be complete if we show $\lambda_{t-1} = \lambda_t + \lambda'_t$.

We have $\lambda_0 = \lambda_1 + \lambda'_1$, for this is simply another way to write the hypothesis $b = r+r'$.

Now we assume $\lambda_{t-2} = \lambda_{t-1} + \lambda'_{t-1}$ and show $\lambda_{t-1} = \lambda_t + \lambda'_t$. To do so we use the following basic consequences of Theorem 1.1:

$$\lambda_t = \frac{\lambda_{t-1}(k-t+1)}{v-t+1}, \tag{1}$$

$$\lambda_{t-2} = \frac{\lambda_{t-1}(v-t+2)}{k-t+2}, \tag{2}$$

$$\lambda'_t = \frac{\lambda'_{t-1}(k-t+2)}{v-t+1}. \tag{3}$$

Using (1), (2), and (3) together with the induction hypothesis, we have

$$\begin{aligned}\lambda_t + \lambda'_t &= \frac{\lambda_{t-1}(k-t+1) + \lambda'_{t-1}(k-t+2)}{v-t+1} \\ &= \frac{(k-t+2)(\lambda_{t-1} + \lambda'_{t-1}) - \lambda_{t-1}}{v-t+1} \\ &= \frac{(k-t+2)\lambda_{t-2} - \lambda_{t-1}}{v-t+1} = \lambda_{t-1}.\end{aligned}$$

The conditions of the theorem seem rather restrictive, but, in fact, there are many applications of the theorem. All of the designs used as hypotheses in applications 1-3 below can be found in [4]. To the author's knowledge, the block designs constructed here are all new.

Applications to block designs

1. There are (15, 35, 14, 6, 5) and (15, 15, 7, 7, 3) block designs. Thus, by taking multiples of the second design, there is a (15, 45, 21, 7, 9) block design. Hence, by the theorem, there is a (16, 80, 35, 7, 14) block design.

2. (15, 42, 14, 5, 4) and (15, 35, 14, 6, 5) — that is (15, 70, 28, 6, 10) — give us the block design (16, 112, 42, 6, 14).

3. (31, 93, 15, 5, 2) and (31, 31, 6, 6, 1) — that is (31, 401, 78, 6, 13) — give the block design (32, 494, 93, 6, 15).

Applications to 3-designs

4. Since there are 3-designs with parameters (10, 30, 12, 4, 1) [6] and (10, 36, 18, 5, 3) [7], there is a 3-design with parameters (11, 66, 30, 5, 4). This is a new construction of this design, but its existence is well known since Witt [9] showed there is even an (11, 66, 30, 5, 1) 4-design.

5-7. Let

$$\pi_1 = (14, 91, 26, 4, 1), \qquad \pi_2 = (14, 182, 65, 5, 5),$$

$$\pi_3 = (14, 91, 39, 6, 5), \qquad \pi_4 = (14, 52, 26, 7, 5).$$

3-designs with parameters π_1 [5] and π_3 [6] exist. π_2 and π_4 are unknown. Hence, if there is a design with parameters π_2, then combining π_1 and π_2 there is a (15, 273, 91, 5, 6) 3-design, and combining π_2 and 3 copies of π_3 there is a (15, 455, 182, 6, 20) 3-design. Also, if there is a 3-design with parameters twice π_4, then — combining π_3 and twice π_4 — there is a 3-design with parameters (15, 195, 91, 7, 15).

8. The 3-design with parameters (16, 30, 15, 8, 3) exists [6], so there is a (16, 90, 45, 8, 9) 3-design. Thus, if the unknown 3-design (16, 80, 35, 7, 5) exists, then the 3-design (17, 170, 80, 8, 14) exists.

9. The 3-design (17, 68, 20, 5, 1) exists [6]. So, if the unknown 3-design (17, 136, 48, 6, 4) exists, then the unknown 3-design (18, 204, 68, 6, 5) exists.

An application to 4-designs

10. The 4-design (23, 253, 77, 7, 1) exists [6]. Thus if (23, 506, 176, 8, 4) exists, we can construct a (24, 759, 253, 8, 5) 4-design. We remark that this latter 4-design is already known, since there is even a (24, 759, 253, 8, 1) 5-design.

THEOREM 2.2. *If there is a* (v, b, r, k, λ) *t-design and a* (k, b', r', l, λ') *t-design and* $t \leqslant l < k$, *then there is a* $(v, bb', rr', l, \lambda\lambda')$ *t-design.*

Proof. Let $(\mathscr{P}, \mathscr{B})$ be a (v, b, r, k, λ) t-design. For each $B \in \mathscr{B}$ let $\mathscr{D}_B$ be a (k, b', r', l, λ') t-design on the points of B. Let

$$\mathscr{D} = \bigcup_{B \in \mathscr{B}} \mathscr{D}_B.$$

Since each block of $\mathscr{B}$ has been replaced by b' blocks, we have bb' blocks in all. Each of the r blocks through a point has been replaced by a design with r' blocks through each point. Hence each point has rr' blocks through it. Any t-points determine λ blocks of $\mathscr{B}$ and each block is replaced by a design in which the t points determine λ' blocks. Hence t points determine $\lambda\lambda'$ blocks in all.

COROLLARY 1. *If there is a (v, b, r, k, λ) t-design and $t \leqslant l < k$, then there is a* $\left(v, b\binom{k}{l}, r\binom{k-1}{l-1}, l, \lambda\binom{k-t}{l-t}\right)$ *t-design.*

Proof. Apply Theorem 2.2 with the second design being the complete $\left(k, \binom{k}{l}, \binom{k-1}{l-1}, l, \binom{k-t}{l-t}\right)$ t-design.

COROLLARY 2. *If there is a (v, b, r, k, λ) t-design and $t < k-1$, then there is a $(v, bk, r(k-1), k-1, \lambda(k-t))$ t-design.*

Proof. In Corollary 1 take $l = k-1$.

3. Families of designs. We will denote families of designs by a numeral and a letter. Thus (2a) is our first family of 2-designs, (3c) the third family of 3-designs, etc.

THEOREM 3.1. *If $4m+3$ and $8m+7$ are prime powers, then there is a block design with parameters*

$$\text{(2a)} \qquad (8m+7, 32m^2+52m+21, 8m^2+10m+3, 2m+1, 2m^2+m).$$

Proof. Apply Theorem 2.2 to designs belonging to the following family shown by Bose [2] to exist whenever $4m+3$ is prime:

$$\text{(B1)} \qquad (4m+3, 4m+3, 2m+1, 2m+1, m).$$

THEOREM 3.2. *If $4m+3$ is a prime power, then there is a block design with parameters*

$$\text{(2b)} \qquad (4m+3, 16m^2+20m+6, 4m^2+2m, m, m^2-m).$$

Proof. Apply Theorem 2.2 to designs belonging to the family (B1) and to the following family of Bose [2] which also exists when $4m+3$ is prime:

$$\text{(B2)} \qquad (2m+1, 4m+2, 2m, m, m-1).$$

Theorem 2.2 is also applicable to 3-designs.

THEOREM 3.3. *If $4m+3$ is a prime power, then a 3-design exists with parameters*

$$(4m+4,\ 64m^2+80m+24,\ 16m^2+20m+6,\ m+1,\ m^2-m). \tag{3a}$$

Proof. Apply Theorem 2.2 to 3-designs belonging to the following families of Sprott [7]:

$$(4m+4,\ 8m+6,\ 4m+3,\ 2m+2,\ m), \tag{S1}$$

$$(2m+2,\ 8m+4,\ 4m+2,\ m+1,\ m-1). \tag{S2}$$

Both designs exist whenever $4m+3$ is prime.

Using Corollary 2, we will derive many new families of designs, but first we list a few particular results of some interest in themselves. The following 2-, 3- and 5-designs are new:

1. Since there is a (11, 11, 5, 5, 2) block design, there is a (11, 55, 20, 4, 6) block design.

2. Since there is a (12, 22, 11, 6, 2) 3-design, there is a (12, 132, 55, 5, 6) 3-design.

3. If there is a (12, 66, 33, 6, 2) 4-design, then there is a (12, 396, 165, 5, 4) 4-design.

4. Since there is a (24, 759, 253, 8, 1) 5-design [9], there is a (24, 6072, 1771, 7, 3) 5-design.

THEOREM 3.4. *If m is a prime power, then block designs exist with parameters*

$$(m^2,\ m^3+m^2,\ m^2-1,\ m-1,\ m-2), \tag{2c}$$

$$(m^2+m+1,\ m^3+2m^2+2m+1,\ m^2+m,\ m,\ m-1). \tag{2d}$$

Proof. Apply Corollary 2 to the families of affine planes and projective planes.

For example, by Theorem 3.4 with $m = 4$, we conclude that there are block designs with parameters (16, 80, 15, 3, 2) and (21, 105, 20, 4, 3). The latter design is new.

THEOREM 3.5. *If $4m+3$ is a prime power, then block designs exist with parameters*

$$(4m+3,\ 8m^2+10m+3,\ 4m^2+2m,\ 2m,\ 2m^2-m), \tag{2e}$$

$$(2m+2,\ 4m^2+6m+2,\ 2m^2+m,\ m,\ m^2-m), \tag{2f}$$

$$(2m+1,\ 4m^2+2m,\ 2m^2-2m,\ m-1,\ m^2-3m+2), \tag{2g}$$

and 3-designs exist with parameters

$$(4m+4,\ 16m^2+28m+12,\ 8m^2+10m+3,\ 2m+1,\ 2m^2-m), \tag{3b}$$

$$(2m+2,\ 8m^2+12m+4,\ 4m^2+2m,\ m,\ m^2-3m+2). \tag{3c}$$

Proof. Apply Corollary 2 to the families (B1), (B3), (B2), (S1), and (S2), respectively, where (B3) is the following family of Bose [2]:

$$(2m+2,\ 4m+2,\ 2m+1,\ m+1,\ m). \tag{B3}$$

For example, taking $m = 4$, we conclude that there are designs belonging to (2e) and (3b) with the following parameters:

1. (19, 171, 72, 8, 28) 2-design,
2. (20, 380, 171, 9, 28) 3-design.

Since the parameters are rather involved, we give the following families in the short form — mentioning only v, k and λ:

THEOREM 3.6. *Assume $n \geqslant 4$. Then*

1. *there is a 3-design with parameters*

$$\big(2^n,\ 2^{n-1}-1,\ (2^{n-1}-3)(2^{n-2}-1)\big); \tag{3d}$$

2. *there is a 4-design with parameters*

$$\big(2^n+1,\ 2^{n-1}-1,\ (2^{n-1}-3)(2^{n-2}-1)(2^{n-1}-4)\big); \tag{4a}$$

3. *there is a 5-design with parameters*

$$\big(2^n+2,\ 2^{n-1},\ (2^{n-1}-3)(2^{n-2}-1)(2^{n-1}-4)\big). \tag{5a}$$

Proof. Apply Corollary 2 to the following Alltop's [1] families:

$$(2^n,\ 2^{n-1},\ 2^{n-2}-1), \tag{A1}$$

$$\big(2^n+1,\ 2^{n-1},\ (2^{n-1}-3)(2^{n-2}-1)\big), \tag{A2}$$

$$\big(2^n+2,\ 2^{n-1}+1,\ (2^{n-1}-3)(2^{n-2}-1)\big). \tag{A3}$$

We mention in passing that (A3) and (5a) are the only known families of 5-designs.

THEOREM 3.7. *Assume $4m+3$ is a prime power.*

1. *If $4m+4 = 2^n$ for some $n \geqslant 5$, then there is a 3-design with parameters*

$$(8m+8,\ 2m+2,\ 2m^2+m). \tag{3e}$$

2. *If $2m+2 = 2^n$ for some $n \geqslant 5$, then there is a 3-design with parameters*

$$(4m+4,\ m,\ m^3-3m^2+2m). \tag{3f}$$

3. *If $2m+2 = 2^n$ for some $n \geqslant 5$, then there is a 3-design with parameters*

(3g) $$(4m+4,\ m+1,\ m^2-m).$$

Proof. By (A1) there is a 3-design with parameters

$$(2^{n+1}, 2^n, 2^{n-1}-1).$$

By (S1) there is a 3-design with parameters

$$(4m+4, 2m+2, m).$$

Since $2^{n+1} = 8m+8$ and $2^{n-1}-1 = 2m+1$, we conclude by composition Theorem 2.2 that there is an $(8m+8, 2m+2, 2m^2+m)$ 3-design. Parts 2 and 3 of the theorem are proved in the same manner, composing (A1) with (3c) and (S2), respectively.

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