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A COUNTEREXAMPLE IN NON-METRIC CONTINUA THEORY

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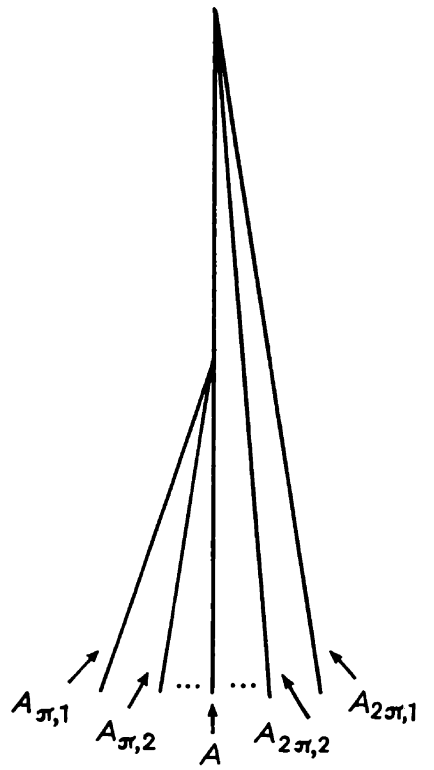
1. The word *continuum* will be used here to denote a compact connected Hausdorff space. The theory of continua was originally developed for metric spaces only, but many important results have been generalized into the Hausdorff setting. Notable examples are Gòrdh's [2] generalization of Kuratowski's (see [3], Section 48) and Thomas' [10] structure theory for irreducible continua (see also Mańka [6]) and Maćkowiak's [3] generalization of Mańka's [5] fixed point theorem for λ-dendroids. A good bit of the similarity between metric and non-metric continua stems from the fact that what may be regarded as the fundamental theorem of continua theory, that the closure of any component of an open set meets the boundary, remains true in the more general setting. There are, however, substantial differences between the two theories. A striking example is the work of Mardešić [7], [8] and Treybig [11], [12] on local connectedness, arcwise connectedness, and the Hahn–Mazurkiewicz theorem for non-metric continua. An important recent example is Bellamy's [1] construction of an indecomposable continuum with one (or two) composant(s) (see also [6]).

In [9] I proved that a metric continuum X is locally connected if and only if every open and connected subset of X is continuum-wise connected. In that paper I presented an example which purported to show that this result fails for Hausdorff continua. It has been pointed out to me by Lewis Lum that this example does not work. If one removes a straight arc in the space joining the two points $(c, 0)$ and $(0, c)$ where $0 < c < 1$ (see [9], p. 84), one obtains an open set which is connected but not continuum-wise connected. I wish to thank Professor Lum for bringing this problem to my attention. In this note I give another example which does work. It is a continuum with the property that each of its connected open subsets is arcwise connected, but which fails to be locally connected. I should also point out, as I did not in the previous paper, that the space has the property that each of its points is either an endpoint or a separating point. The latter property characterizes dendrites among metric continua, implying both local connectedness and acyclicity. Previous examples of the failure of this characterization in the non-metric setting have been given, but, as far as I know, the space described in [9] is the first non-locally connected example. The so-called "comb space" over a circle is a cyclic example.

2. The example I wish to describe now is made up of a collection of straight arcs in 3-dimensional space with cylindrical coordinates (r, θ, z). Let A denote the closed segment of the z-axis joining the points (0, 0, 0) and



$(0, 0, 2\pi)$. Points $(0, 0, t)$ in A will be denoted shortly by t. For each $t \in A - \{0\}$ and for each natural number n, let $A'_{t,n}$ denote the closed straight arc joining the points $(0, 0, t)$ and $(1/n, t, 0)$ (see drawing) and let $A_{t,n} = A'_{t,n} - \{t\}$. Let

$$\mathscr{A}' = \{A'_{t,n}\colon\ n = 1, 2, 3, \ldots \text{ and } t \in A - \{0\}\}$$

and

$$\mathscr{A} = \{A_{t,n}\colon\ n = 1, 2, 3, \ldots \text{ and } t \in A - \{0\}\}.$$

Let $X = A \cup \bigcup \mathscr{A}$ be topologized as follows. If $x \in X - A$, then x is an element of some $A_{t,n} \in \mathscr{A}$. A neighborhood base at x will consist of all open subintervals (half open if the z-coordinate of x is 0) of $A_{t,n}$ containing x. Now suppose that $t \in A$. Let a and b be real numbers such that $a < t < b$ and let $\mathscr{F}$ be a finite subset of $\mathscr{A}'$ such that $t \notin \bigcup \mathscr{F}$. Then a, b, and $\mathscr{F}$ define the following neighborhood of t:

$$\{(r, \theta, z) \in X\colon\ a < z < b\} - \bigcup \mathscr{F}.$$

The collection of all such neighborhoods defines a neighborhood base at t.

It is straightforward to verify that X with the topology just defined is compact and Hausdorff and fails to be locally connected. Now suppose that U is a connected open subset of X. We wish to show that U is arcwise connected. If $U \subset X - A$, then U must be a subset of some particular $A_{t,n}$, since these sets are clopen in $X - A$. Therefore, U must be an open interval and we are done. Next suppose that $U \cap A \neq \emptyset$. If $x \in U \cap A_{t,n}$ for some natural number n and $t \in A$, then an argument similar to the one just given, but a little more involved, will show that the entire closed arc in $A'_{t,n}$ from x to t must lie in U. Thus each point of $U \cap (X - A)$ lies in an arc contained in U and meeting A. It remains only to show that $U \cap A$ is a subinterval of A. Just suppose that this were not true. Then there must be points $t_1, t_2, t_3 \in A$ such that $t_1 < t_2 < t_3$, $t_1, t_3 \in U$ and $t_2 \notin U$. Let $F = X - U$. Since each of the sets $A_{t_2,n}$ is clopen in $X - \{t_2\}$, they must all be contained in F. Since F is closed, each point in the closure of $\bigcup \{A_{t_2,n}: n = 1, 2, 3, \ldots\}$ must lie in F. But this includes all points $t \in A$ such that $t < t_2$, in particular the point t_1. This contradiction shows that $U \cap A$ must be an interval and establishes the desired claim for U.

REFERENCES

[1] D. Bellamy, *Indecomposable continua with one and two composants*, Fundamenta Mathematicae 101 (1978), p. 129-134.

[2] G. R. Gordh, Jr., *Monotone decompositions of irreducible Hausdorff continua*, Pacific Journal of Mathematics 36 (1971), p. 647-658.

[3] K. Kuratowski, *Topology. II*, New York–Warszawa 1968.

[4] T. Maćkowiak, *Fixed point property for λ-dendroids*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques 26 (1978), p. 61-64.

[5] R. Mańka, *Association and fixed points*, Fundamenta Mathematicae 91 (1976), p. 105-121.

[6] — *On the characterization by Kuratowski of irreducible continua*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques 26 (1978), p. 49-55.

[7] S. Mardešić, *Mapping ordered continua onto product spaces*, Glasnik Matematičko-Fizički i Astronomski, Društvo Matematičara i Fizičara N.R. Hrvatske, Serija II, 15 (1960), p. 85-89.

[8] — *On the Hahn–Mazurkiewicz theorem in non-metric spaces*, Proceedings of the American Mathematical Society 11 (1960), p. 929-937.

[9] L. Mohler, *A characterization of local connectedness for generalized continua*, Colloquium Mathematicum 21 (1970), p. 81-85.

[10] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Dissertationes Mathematicae 50 (1966).

[11] L. B. Treybig, *Concerning continuous images of compact ordered spaces*, Proceedings of the American Mathematical Society 15 (1964), p. 866-871.

[12] — *Concerning continua which are the continuous images of compact ordered spaces*, Duke Mathematical Journal 32 (1965), p. 417-422.

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