COLLOQUIUM MATHEMATICUM

VOL. XXXVIII 1978 FASC. 2

ASSERTION Q DISTINGUISHES TOPOLOGICALLY ω^* AND $\mathfrak{m}^*$ WHEN $\mathfrak{m}$ REGULAR AND $\mathfrak{m} > \omega$

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We consider the following question: can ω^* be homeomorphic to $\mathfrak{m}^*$ for $\mathfrak{m} > \omega$, ω denoting the set of non-negative integers? Here $\mathfrak{m}^*$ stands for $\beta\mathfrak{m} - \mathfrak{m}$, where $\beta\mathfrak{m}$ is the Čech-Stone compactification of the set $\mathfrak{m}$ with the discrete topology. Only the case $2^\omega = 2^\mathfrak{m}$ is of interest, since only in that case the weights of ω^* and $\mathfrak{m}^*$ are equal. We answer the question for regular $\mathfrak{m}$ negatively in ZFC + non CH + Q, where CH stands for the continuum hypothesis and Q is the following assertion (Rothberger [4]):

For each family $\mathscr{K}$ of less than 2^ω functions $f\colon \omega \to \omega$ there exists a function $g\colon \omega \to \omega$ such that for each $f \in \mathscr{K}$ the set $\{n \in \omega\colon f(n) \geqslant g(n)\}$ is finite.

Assertion Q is known to be a theorem in the theory ZFC + non CH + + Martin's Axiom (Kunen and Tall [2]) which is consistent if ZFC is (Martin and Solovay [3]); in that theory, $2^\mathfrak{m} = 2^\omega$ if $\omega \leqslant \mathfrak{m} < 2^\omega$ (see papers [2] and [3]).

The problem to distinguish ω^* and $\mathfrak{m}^*$ by means of ZFC axioms only seems to be open. **(1021)**

A *free ultrafilter* in a set is a maximal filter on that set not containing finite subsets.

The space $\mathfrak{m}^*$ consists of all free ultrafilters on $\mathfrak{m}$; the topology on $\mathfrak{m}^*$ is generated by sets $U^* = \{p\colon U \in p,\ p \in \mathfrak{m}^*\}$, where U is an infinite subset of $\mathfrak{m}$.

An uncountable cardinal $\varkappa$ is called *measurable* if there exists a free ultrafilter q on $\varkappa$ such that $\bigcap \mathscr{G} \in q$ for each countable subfamily $\mathscr{G}$ of q.

An ultrafilter $q \in \mathfrak{m}^*$ is called a *P-ultrafilter* if for every sequence consisting of members of q there exists a member U of q such that $U - V$ is finite for each V from that sequence.

LEMMA. *Let $\mathfrak{m}$ be non-measurable and let q be a P-ultrafilter on $\mathfrak{m}$. Then there exists a countable subset a of $\mathfrak{m}$ such that $a \in q$.*

Proof. Since $\mathfrak{m}$ is non-measurable, there exists a sequence $\{a_i: i \in \omega\}$ such that

$$a_i \in q, \ a_i \supseteq a_{i+1}, \quad \text{and} \quad \bigcap_{i \in \omega} a_i = 0.$$

Since q is a P-ultrafilter, there exists an infinite a, $a \in q$, such that $a - a_i$ is finite for all i and $a \subseteq a_1$. Hence the sets $a \cap (a_i - a_{i+1})$ are finite. But a is infinite and equal to

$$\bigcup_{i \in \omega} \big(a \cap (a_i - a_{i+1})\big),$$

thus a is countable.

Let $V = \{\mathfrak{m} - a: a \subseteq \mathfrak{m}, \operatorname{card} a \leqslant \omega\}$. Since $cf(\mathfrak{m}) > \omega$, $\mathfrak{m}$ being regular, there exists a filterbase of cardinality $\mathfrak{m}$ such that the filter generated by that filterbase contains V.

For instance, the set

$$\mathscr{F} = \{\mathfrak{m} - a: \ a \text{ is an ordinal}, \ a < \mathfrak{m}\}$$

in such a filterbase.

THEOREM (ZFC + non CH + Q, Szymański [6], Solomon [5] for $\mathfrak{m} = \omega_1$). *Let $\mathfrak{m}$ be a regular cardinal, $\mathfrak{m} < 2^\omega$, and let $\{a_\xi: \xi < \mathfrak{m}\}$ be a family of closed-open subsets of ω^* linearly ordered by inclusion. Then there exists a P-ultrafilter q on ω such that*

$$q \in \bigcap_{\xi < \mathfrak{m}} a_\xi.$$

THEOREM (ZFC + non CH + Q). *ω^* is not homeomorphic to $\mathfrak{m}^*$ whenever $\mathfrak{m}$ is regular and such that $\mathfrak{m} > \omega$.*

Proof. Only the case $\mathfrak{m} < 2^\omega$ requires a proof. Let f be a homeomorphism from ω^* onto $\mathfrak{m}^*$ and let

$$\mathscr{F}^* = \{a^*: a \in \mathscr{F}\}, \quad \text{where } a^* = \mathrm{cl}_{\beta\mathfrak{m}} a - \mathfrak{m}.$$

The family $\mathscr{F}$ is linearly ordered by inclusion and $\operatorname{card} \mathscr{F}^* = \mathfrak{m}$. The family

$$\mathscr{G} = \{f^{-1}(a^*): \ a \in \mathscr{F}\}$$

is also linearly ordered by inclusion, and each element of $\mathscr{G}$ is a closed-open subset of ω^*.

From the above-quoted result of Szymański we infer that there exists a P-ultrafilter $q \in \omega^*$ such that $q \in \bigcap \mathscr{G}$. The ultrafilter $f(q)$ is a P-ultrafilter on $\mathfrak{m}$, f being a homeomorphism. We can see that $f(q) \in \bigcap \mathscr{F}^*$. This means that each element of $\mathscr{F}$, and, therefore, each element of V, belongs to $f(q)$. The ultrafilter $f(q)$ is not a P-ultrafilter on $\mathfrak{m}$. In fact,

$\mathfrak{m}$ is non-measurable by Ulam's theorem (Ulam [7], and Jech [1], p. 167), $\mathfrak{m}$ being not greater than 2^{ω}.

In other case, by the Lemma, a countable a would exist with $a \in f(q)$; however, we know that $\mathfrak{m} - a$ belongs to V and, therefore, to $f(q)$; a contradiction.

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Reçu par la Rédaction le 29. 4. 1976