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METRIC BETWEENNESS IN NORMED LINEAR SPACES

BY

F. A. TORANZOS (BUENOS AIRES)

1. Introduction. Let $(E; |\ |)$ be a real normed linear space and θ its origin. Denote $U(x, a) = \{t: |x-t| \leqslant a\}$; $S(x, a) = \{t: |x-t| = a\}$; in particular, $U = U(\theta, 1)$ and $S = S(\theta, 1)$. The open linear segment (resp., closed linear segment) with extremes x and y is denoted by $(x; y)$ (resp., $[x; y]$). If x, y and z are points of E, we say that z *is between* x *and* y iff $|x-z|+|z-y| = |x-y|$. $B(x, y)$ is the set of points between x and y. From the properties of the norm it follows that $[x; y] \subset B(x, y)$ for every pair $\{x; y\}$. The pair $\{x; y\}$ is *tense* (notation: $x\tau y$) iff $[x; y] = B(x, y)$, that is, if metric and algebraic betweenness coincide for this pair.

The aim of this note is to describe in terms of the geometry of U, for any x in E, the set of all y such that $x\tau y$ (see Theorem 2.8). An easy consequence of such description is a characterization of tense spaces, i.e. normed linear spaces such that any pair of points is tense (see below, Theorem 3.2) This type of spaces is frequently used in papers on Distance Geometry (see, for instance, [1] and [6]).

2. The relation τ. If $A \subset E$, $t \in E$, $\lambda \in R$, write $\lambda A = [\lambda x : x \in A]$ and $A+t = \{x+t: x \in A\}$.

LEMMA 2.1. *If x, y and z are points of E and $\lambda \in R$, then*

(i) $B(\lambda x, \lambda y) = \lambda B(x, y)$.

(ii) $B(x+z, y+z) = B(x, y)+z$.

Proof. If $t \in B(x, y)$, then

(i) $|\lambda x-\lambda t|+|\lambda t-\lambda z| = |\lambda|(|x-t|+|t-z|) = |\lambda|\,|x-z| = |\lambda x-\lambda z|$.

Hence $\lambda t \in B(\lambda x, \lambda y)$. The converse inclusion follows in the same way.

(ii)
$$|(x+z)-(t+z)|+|(t+z)-(y+z)| = |x-t|+|t-y| = |x-y| = |(x-z)+(y-z)|.$$

Hence $t+z \in B(x+z, y+z)$. The converse inclusion follows clearly.

PROPOSITION 2.2. *The relation τ satisfies the properties*

(i) $\forall x \in E$, $x\tau x$.

(ii) $x\tau y$ *implies* $y\tau x$.

(iii) $\forall \lambda \in R$, $x\tau y$ *implies* $(\lambda x)\tau(\lambda y)$.

(iv) $\forall z \in E$, $x\tau y$ *implies* $(x+z)\tau(y+z)$.

Proof. (i) and (ii) are immediate from the definition of τ.

(iii) From 2.1 (i) it follows that

$$B(\lambda x, \lambda y) = \lambda B(x, y) = \lambda |x; y| = |\lambda x; \lambda y|.$$

(iv) From 2.1 (ii) it follows that

$$B(x+z, y+z) = B(x, y)+z = |x; y|+z = |x+z; y+z|.$$

Write $T(x) = \{y: x\tau y\}$ and, in particular, $T_0 = T(\theta)$.

LEMMA 2.3. *For any* $x \in E$, $T(x) = T_0 + x$.

Proof. Inmediate from 2.2 (iv).

LEMMA 2.4. $T(x)$ *is a union of lines through* x.

Proof. In view of 2.3 it is enough to show that T_0 is a union of lines through θ. But if $y \in T_0$ and $\lambda \in R$, then it follows from 2.2 (iii) that

$$\lambda y \in T(\lambda\theta) = T_0.$$

LEMMA 2.5. *If* $t \in B(x, y)$, *then* $[t; x] \subset B(x, y)$.

Proof. This is a particular case of the transitivity of metric betweenness. For the general statement and proof, see [2], p. 33, theorem 12.1. (3).

LEMMA 2.6. *The following statements are aquivalent*:

(i) $x\tau y$,

(ii) $|x-z| = |z-y| = (1/2)|x-y|$ *implies* $z = (1/2)(x+y)$.

Proof. (i) $\to$ (ii) Trivial.

(ii) $\to$ (i). Let $a = |x-y|$ and $t \in B(x, y)$. We can assume without loss of generality that $|x-t| = \beta > a/2 > a-\beta = |y-t|$. Set $z = (a/2\beta)t + (2\beta - a/2\beta)x$. Hence $|x-z| = a/2$, and since $z \in [x; t]$, it follows from 2.5 that $|y-z| = a/2$. By (ii), $z = (1/2)(x+y)$ and $t = (a-\beta/a)x + (\beta/a)y \in [x; y]$.

If F is a closed convex set, an *extreme point* of F is a point $x \in F$ such that there is no open segment $(y; z) \subset F$ containing x. The set of all extreme points of F is denoted by $\mathrm{ex}\, F$. Clearly, $\mathrm{ex}\, F \subset \mathrm{bdry}\, F$.

PROPOSITION 2.7. $T_0 \cap S = \mathrm{ex}\, U$.

Proof. (a) $T_0 \cap S \subset \mathrm{ex}\, U$. Take a point x in the first set and assume $x \notin \mathrm{ex}\, U$. Since extremality is preserved under homotetias, $x/2$ is not an extreme point of $U(\theta, 1/2)$, and so there are points v and w in $S(\theta, 1/2)$ such that $x/2 \in (v; w) \subset S(\theta, 1/2)$. By the central symmetry of $S(\theta, 1/2)$, $-x/2 \in (-v; -w) \subset S(\theta, 1/2)$. Applying the translation $T: y \to y+x$ we get $x/2 \in (x-v; x-w) \subset S(x, 1/2)$. The segments $[v; w]$ and $[x-v; x-w]$ are parallel and have a common interior point, namely $x/2$, hence there is a non-degenerate segment $[y; z]$ containing $x/2$ and such that

$$[y; z] \subset [v; w] \cap [x-v; x-w] \subset S(\theta, 1/2) \cap S(x, 1/2).$$

As a consequence, there is $t \neq x/2$ and $t \in S(\theta, 1/2) \cap S(x, 1/2)$. In virtue of Lemma 2.6, $x \notin T_0$. A contradiction.

(b) ex $U \subset T_0 \cap S$. Suppose $x \in S$ and $x \notin T_0$. By 2.6 there is $t \neq x/2$ such that $|t| = |t - x| = (1/2)|x|$. Set $z = x - t$. Then $|z| = |z - x| = (1/2)|x|$. But $x/2 \in (t; z) \subset S(\theta, 1/2) \cap S(x, 1/2)$ and, consequently, $x/2 \notin \operatorname{ex} U(\theta, 1/2)$. From a previous remark it follows that $x \notin \operatorname{ex} U$.

THEOREM 2.8. *$T_0 = \{\lambda x : \lambda \in R$ and $x \in \operatorname{ex} U\}$.*

Proof. Follows immediately from 2.7 and 2.4.

COROLLARY 2.9. *Let x and y be points of E, $a = |x - y|$. Then $x\tau y$ iff $y \in \operatorname{ex} U(x, a)$.*

Proof. Assume $y \in T(x)$. Then $y - x \in T_0$ and, by 2.8, $t = (1/a)(y - x) \in \operatorname{ex} U$. By a previous remark $y - x \in \operatorname{ex} U(\theta, a)$ and $y \in \operatorname{ex} U(x, a)$. The converse implication follows in the same way.

3. Tense spaces. Fréchet [6] defined a normed linear space E to be tense (espace tendu) if metric and algebraic betweenness coincide everywhere, that is, using our notation, if $T_0 = E$. A closed convex set K is rotund if, for any support hyperplane H, the intersection of H with K is a single point. A normed linear space is rotund if its unit ball is rotund. For further information regarding rotund spaces the reader is referred to [5]. The following characterizations of rotundity are very simple and the proofs are left to the reader.

LEMMA 3.1. *If K is a closed convex set, then the following statements are equivalent:*

(i) *K is rotund.*

(ii) *There is no nondegenerate segment in* bdry K.

(iii) Bdry $K = \operatorname{ex} K$.

THEOREM 3.2. *A normed linear space is tense iff it is rotund.*

Proof. Follows easily from 2.8 and 3.1 (iii).

The preceding result was previously proved in [3], [4], [7] and [9], and in a somewhat more restrictive environment in [8]. In a way, our main result 2.8 complements the results obtained in [8].

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UNIVERSIDAD DE BUENOS AIRES

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