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SOME REMARKS ON ORTHOMORPHISMS

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Orthomorphisms were first introduced in [14] by Nakano under the name *dilatators*, and since then they have been very useful in many contexts. Their remarkable properties have been studied by many authors (see [5], [6], [8], [11], [12], [14], [16], and [18]). However, the basic results of these works are based upon the representations of Riesz spaces by function spaces. In this note we shall show how to derive the fundamental properties of orthomorphisms directly from the lattice properties of Riesz spaces. The main tool for this approach will be a result expressing the lattice operations on the Riesz space of order bounded transformations in terms of components (Theorem 1.3 below).

For terminology and fundamental concepts of Riesz spaces not explained below we refer the reader to [3] and [10]. All Riesz spaces under consideration here are assumed to be Archimedean.

1. The lattice structure of orthomorphisms. We start with the definition of a positive orthomorphism.

Definition 1.1. Let L be an Archimedean Riesz space. A positive (linear) operator $\pi\colon L \to L$ is called a *positive orthomorphism* whenever $u \wedge v = 0$ implies $\pi(u) \wedge v = 0$.

Clearly, if $\pi\colon L \to L$ is a positive orthomorphism, then $u \wedge v = 0$ implies $\pi(u) \wedge \pi(v) = 0$. That is, every positive orthomorphism is a Riesz homomorphism. Also, the sum of two positive orthomorphisms is a positive orthomorphism, and any positive operator dominated by a positive orthomorphism is likewise a positive orthomorphism.

Some elementary characterizations of positive orthomorphisms are included in the next theorem whose proof is trivial.

THEOREM 1.1. *Let $\pi\colon L \to L$ be a positive operator. Then the following statements are equivalent:*

1. *π is a positive orthomorphism.*
2. *For each $u \in L$ we have $\pi(u) \in B_u$ (the band generated by u in L).*
3. *For each band B of L we have $\pi(B) \subseteq B$.*

In [5] and [8] it was shown via the representation theorems that every positive orthomorphism is order continuous. An elementary proof of this result appeared in [9]. Since this property is important for our approach, for completeness another elementary proof is presented below.

THEOREM 1.2. *Every positive orthomorphism is order continuous.*

Proof. Let $\pi: L \to L$ be a positive orthomorphism and let $u_a \downarrow 0$ in L. We can assume that $0 \leqslant u_a \leqslant u$ holds for all a and some $u \in L$. Now let $w \in L$ satisfy $0 \leqslant w \leqslant \pi(u_a) \leqslant \pi(u)$ for each a, and then fix $\varepsilon > 0$.

Since $(u_a - \varepsilon u)^+ \wedge (u_a - \varepsilon u)^- = 0$, it follows that

$$0 \leqslant [w - \varepsilon\pi(u)]^+ \wedge (u_a - \varepsilon u)^- \leqslant \pi\big((u_a - \varepsilon u)^+\big) \wedge (u_a - \varepsilon u)^- = 0,$$

and so

$$0 = [w - \varepsilon\pi(u)]^+ \wedge (u_a - \varepsilon u)^- \uparrow_a [w - \varepsilon\pi(u)]^+ \wedge \varepsilon u .$$

Thus, $[w - \varepsilon\pi(u)]^+ \wedge u = 0$ for each $\varepsilon > 0$, which implies (since L is Archimedean) $w \wedge u = 0$. Consequently, $w = \pi(u) \wedge w = 0$. Therefore, $\pi(u_a) \downarrow 0$ holds in L, as required.

An operator $\pi: L \to L$ is said to be an *orthomorphism* if there exist two positive orthomorphisms π_1, π_2 such that $\pi = \pi_1 - \pi_2$. By Theorem 1.2, every orthomorphism π is an order continuous operator, i.e., if $u_a \xrightarrow{(o)} u$, then $\pi(u_a) \xrightarrow{(o)} \pi(u)$. The collection of all orthomorphisms on L is denoted by $\mathrm{Orth}(L)$. Clearly, $\mathrm{Orth}(L)$ is a (real) vector space. Moreover, if $\pi \geqslant \sigma$ in $\mathrm{Orth}(L)$ means $\pi(u) \geqslant \sigma(u)$ for each $u \in L^+$, then $\geqslant$ is an order relation under which $\mathrm{Orth}(L)$ is a partially ordered vector space. The positive cone of $\mathrm{Orth}(L)$ consists precisely of all positive orthomorphisms on L. Indeed, if the orthomorphism $\pi = \pi_1 - \pi_2$ satisfies $\pi \geqslant 0$, and $u \wedge v = 0$ holds, then

$$0 \leqslant \pi(u) \wedge v \leqslant \pi_1(u) \wedge v + \pi_2(u) \wedge v = 0,$$

so that π is a positive orthomorphism.

In actuality, $\mathrm{Orth}(L)$ is known to be an f-algebra with the identity operator as a multiplicative unit, and where the multiplication is the usual composition of two operators. (A Riesz space L is said to be an *f-algebra* if L is also an algebra whose multiplication satisfies $uv \geqslant 0$ for each $u, v \in L^+$, and if $u \wedge v = 0$ implies $uw \wedge v = wu \wedge v = 0$ for all $w \in L^+$.)

If M is a Dedekind complete Riesz space, then $\mathscr{L}_b(L, M)$ denotes the (Dedekind complete) Riesz space of all order bounded operators from L into M. As usual, we write $\mathscr{L}_b(M)$ for $\mathscr{L}_b(M, M)$.

For the discussion ahead assume at the beginning that L is a Dedekind complete Riesz space. If π_1 and π_2 are two positive orthomorphisms on L, then from $0 \leqslant \pi_1 \vee \pi_2 \leqslant \pi_1 + \pi_2$ and $0 \leqslant \pi_1 \wedge \pi_2 \leqslant \pi_1$ it follows that $\pi_1 \vee \pi_2$ and $\pi_1 \wedge \pi_2$ (taken, of course, in $\mathscr{L}_b(L)$) are both positive orthomorphisms on L. Now, if $\pi = \pi_1 - \pi_2$ and $\sigma = \sigma_1 - \sigma_2$ are two orthomorphisms on L, then the identity

$$\pi \vee \sigma = (\pi_1 + \sigma_2) \vee (\sigma_1 + \pi_2) - (\pi_2 + \sigma_2)$$

shows that $\pi \vee \sigma \in \mathrm{Orth}(L)$. Hence $\mathrm{Orth}(L)$ is a Riesz subspace of $\mathscr{L}_b(L)$. It is now easy to see that $\mathrm{Orth}(L)$ is a band of $\mathscr{L}_b(L)$ containing the identity operator I. Hence $B_I \subseteq \mathrm{Orth}(L)$, where B_I is the band generated by I in $\mathscr{L}_b(L)$. In actuality, it is known that $\mathrm{Orth}(L) = B_I$ (see Theorem 1.4 below).

Recall that if M is Dedekind complete and $T, S \in \mathscr{L}_b(L, M)$, then

$$T \vee S(u) = \sup\{T(v) + S(w) \colon v, w \in L^+ \text{ and } v + w = u\}$$

and

$$T \wedge S(u) = \inf\{T(v) + S(w) \colon v, w \in L^+ \text{ and } v + w = u\}$$

hold in $\mathscr{L}_b(L, M)$ for each $u \in L^+$ (see, e.g., [3], Theorem 3.3, p. 20). The next theorem expresses the above sup and inf in terms of disjoint elements and is of some independent interest in its own right.

THEOREM 1.3. *Let L be a Riesz space with the principal projection property, and M a Dedekind complete Riesz space. Then for each pair $T, S \in \mathscr{L}_b(L, M)$ and $u \in L^+$ we have*

$$T \vee S(u) = \sup\{T(v) + S(w) \colon v \wedge w = 0 \text{ and } v + w = u\}$$

and

$$T \wedge S(u) = \inf\{T(v) + S(w) \colon v \wedge w = 0 \text{ and } v + w = u\}.$$

Proof. The first formula follows from the second. Indeed, if the second is true, then

$$\begin{aligned} T \vee S(u) &= -(-T) \wedge (-S)(u) \\ &= -\inf\{-T(v) - S(w) \colon v \wedge w = 0 \text{ and } v + w = u\} \\ &= \sup\{T(v) + S(w) \colon v \wedge w = 0 \text{ and } v + w = u\}. \end{aligned}$$

Also, if the second formula holds when $T \wedge S = 0$, then it is true in general. Indeed, if this is the case, then the identity $(T - T \wedge S) \wedge \wedge (S - T \wedge S) = 0$ implies

$$\begin{aligned} 0 &= \inf\{(T - T \wedge S)(v) + (S - T \wedge S)(w) \colon v \wedge w = 0 \text{ and } v + w = u\} \\ &= \inf\{T(v) - T \wedge S(v) + S(w) - T \wedge S(w) \colon v \wedge w = 0 \text{ and } v + w = u\} \\ &= \inf\{T(v) + S(w) \colon v \wedge w = 0 \text{ and } v + w = u\} - T \wedge S(u). \end{aligned}$$

Consequently, $T \wedge S(u) = \inf\{T(v) + S(w) \colon v \wedge w = 0 \text{ and } v + w = u\}$ holds in this case.

To complete the proof assume $T\wedge S = 0$ in $\mathscr{L}_b(L, M)$, and $u \in L^+$. Put $e = \inf\{T(v)+S(w)\colon v\wedge w = 0$ and $v+w = u\}$ and fix $0<\varepsilon<1$. Now let v be an arbitrary element of L such that $0\leqslant v\leqslant u$. Denote by P the projection of L onto the band generated by $(v-\varepsilon u)^+$ and put $w = P(u)$. Clearly, $w\wedge(u-w) = 0$. Also, the inequality $(v-\varepsilon u)^+\leqslant u$ implies

$$0\leqslant v-v\wedge \varepsilon u = (v-\varepsilon u)^+ = P\big((v-\varepsilon u)^+\big)\leqslant P(u) = w,$$

and thus

$$u-w\leqslant u-v+v\wedge \varepsilon u. \tag{1}$$

On the other hand, $0\leqslant (v-\varepsilon u)^+ = P(v-\varepsilon u) = P(v)-\varepsilon P(u)\leqslant v-\varepsilon w$ implies

$$w\leqslant \frac{1}{\varepsilon}v. \tag{2}$$

Therefore, using (1) and (2), we obtain

$$\begin{aligned}0\leqslant e&\leqslant T(u-w)+S(w)\leqslant T(u-v+v\wedge\varepsilon u)+S(w)\\ &= T(u-v)+T(v\wedge \varepsilon u)+S(w)\\ &\leqslant T(u-v)+\varepsilon T(u)+\frac{1}{\varepsilon}S(v)\leqslant\frac{1}{\varepsilon}T(u-v)+\varepsilon T(u)+\frac{1}{\varepsilon}S(v)\\ &= \frac{1}{\varepsilon}[T(u-v)+S(v)]+\varepsilon T(u)\end{aligned}$$

for all v $(0\leqslant v\leqslant u)$ and $0<\varepsilon<1$. This implies (in view of $T\wedge S = 0$) $0\leqslant e\leqslant \varepsilon T(u)$ for all ε $(0<\varepsilon<1)$, and hence $e = 0$, as required.

Note. A special case of the preceding theorem was stated without proof for linear functionals in [1] (Theorem 5, p. 513), where it was also noted that it holds for regular operators. The referee has informed us that Theorem 1.3 was proved first in Abramovič's dissertation.

It should be also noted that Theorem 1.3 is false without assuming that L has the principal projection property. For instance, if $L = C[0,1]$, $M = \boldsymbol{R}$, $T(u) = u(0)$, $S(u) = u(1)$, then $T\wedge S = 0$, while

$$\begin{aligned}&\sup\{T(v)+S(w)\colon v\wedge w = 0 \text{ and } v+w = 1\}\\ &\qquad = \inf\{T(v)+S(w)\colon v\wedge w = 0 \text{ and } v+w = 1\} = 1.\end{aligned}$$

We are now in the position to show directly that if L is Dedekind complete, then $\mathrm{Orth}(L)$ is a Riesz space under the pointwise lattice operatios.

THEOREM 1.4. *If L is a Dedekind complete Riesz space, then* $\mathrm{Orth}(L)$ *is precisely the band generated by the identity operator in* $\mathscr{L}_b(L)$. *For every pair* $\pi_1, \pi_2 \in \mathrm{Orth}(L)$ *and* $u \in L^+$ *we have*

$$(3) \qquad \pi_1 \vee \pi_2(u) = \pi_1(u) \vee \pi_2(u) \quad \text{and} \quad \pi_1 \wedge \pi_2(u) = \pi_1(u) \wedge \pi_2(u).$$

Moreover, $\mathrm{Orth}(L)$ *with multiplication the composition operation is an Archimedean f-algebra with unit element the identity operator.*

Proof. First we prove the formulas. To this end, let π_1 and π_2 be two orthomorphisms. Replacing π_1 and π_2 by $\pi_1 - \pi_1 \wedge \pi_2$ and $\pi_2 - \pi_1 \wedge \pi_2$ (if necessary), we can assume that π_1 and π_2 are positive orthomorphisms. Let $u \in L^+$. Clearly,

$$\pi_1 \wedge \pi_2(u) \leqslant \pi_1(u) \wedge \pi_2(u).$$

Next note that if $v, w \in L$ satisfy $v \wedge w = 0$, then $\pi_1(v) \wedge w = \pi_2(v) \wedge \wedge w = 0$ must hold; and therefore $\pi_1(v) \wedge \pi_2(w) = \pi_2(v) \wedge \pi_1(w) = 0$ must also hold. Thus, if $v, w \in L$ satisfy $v \wedge w = 0$ and $v + w = u$, then

$$\begin{aligned} \pi_1(u) \wedge \pi_2(u) &= [\pi_1(v) + \pi_1(w)] \wedge [\pi_2(v) + \pi_2(w)] \\ &\leqslant \pi_1(v) \wedge \pi_2(v) + \pi_1(w) \wedge \pi_2(v) + \pi_1(v) \wedge \pi_2(w) + \pi_1(w) \wedge \pi_2(w) \\ &= \pi_1(v) \wedge \pi_2(v) + \pi_1(w) \wedge \pi_2(w) \leqslant \pi_1(v) + \pi_2(w). \end{aligned}$$

From Theorem 1.3 it follows that

$$\pi_1(u) \wedge \pi_2(u) \leqslant \pi_1 \wedge \pi_2(u),$$

and so $\pi_1 \wedge \pi_2(u) = \pi_1(u) \wedge \pi_2(u)$. The other formula follows easily from the identity $\pi_1 \vee \pi_2 = \pi_1 + \pi_2 - \pi_1 \wedge \pi_2$.

Next we shall show that $\mathrm{Orth}(L) = B_I$. We have already seen that $B_I \subseteq \mathrm{Orth}(L)$. For the converse relation, let $0 \leqslant \pi \in \mathrm{Orth}(L)$ and $u \in L^+$. By Theorem 1.1, $\pi(u) \in B_u$, and so

$$\pi \wedge nI(u) = \pi(u) \wedge nu \uparrow \pi(u);$$

that is, $\pi \wedge nI \uparrow \pi$ in $\mathscr{L}_b(L)$. Since $\{\pi \wedge nI\} \subseteq B_I$, we have $\pi \in B_I$, so that $\mathrm{Orth}(L) \subseteq B_I$. Hence $\mathrm{Orth}(L) = B_I$. The last assertion should be now immediate.

Now let us consider an arbitrary Archimedean Riesz space L. The Dedekind completion of L will be denoted by L^δ. If M is a Dedekind complete Riesz space, and $T: L \to M$ is a positive order continuous operator, then it is well known that T can be extended uniquely to a positive order continuous operator T^* from L^δ into M. Specifically,

$$T^*(u) = \sup\{T(w): w \in L \text{ and } w \leqslant u\} = \inf\{T(v): v \in L \text{ and } v \geqslant u\}$$

for each $u \in L^\delta$ (see [3], p. 26).

If $\pi\colon L \to L \subseteq L^\delta$ is a positive orthomorphism, then (by Theorem 1.2) π is order continuous, and hence it has a unique positive extension π^* from L^δ into L^δ. It is easy to see that π^* is likewise a positive orthomorphism on L^δ. From this observation it follows that every orthomorphism π on L extends uniquely to an orthomorphism π^* on L^δ.

It is not difficult to see that $\pi \mapsto \pi^*$ from $\mathrm{Orth}(L)$ into $\mathrm{Orth}(L^\delta)$ is linear and one-to-one. Moreover, $\pi \geqslant 0$ if and only if $\pi^* \geqslant 0$. Therefore, $\mathrm{Orth}(L)$ is a Riesz space, and $\pi \mapsto \pi^*$ is a Riesz isomorphism from $\mathrm{Orth}(L)$ into $\mathrm{Orth}(L^\delta)$. Note that if $\pi_1, \pi_2 \in \mathrm{Orth}(L)$, then (by Theorem 1.4) formulas (3) hold in $\mathrm{Orth}(L)$ for each $u \in L^+$. In other words, if we identify $\mathrm{Orth}(L)$ with its image under $\pi \mapsto \pi^*$ in $\mathrm{Orth}(L^\delta)$, then $\mathrm{Orth}(L)$ is the Riesz subspace of $\mathrm{Orth}(L^\delta)$ consisting of all orthomorphisms on L^δ that leave L invariant. Rephrasing the above we have the following result:

THEOREM 1.5. *If L is an Archimedean Riesz space, then* $\mathrm{Orth}(L)$ *is an Archimedean f-subalgebra (with unit being the identity operator) of* $\mathrm{Orth}(L^\delta)$. *Moreover, if $\pi_1, \pi_2 \in \mathrm{Orth}(L)$ and $u \in L^+$, then formulas* (3) *hold true.*

We continue with an important property of orthomorphisms.

LEMMA 1.1. *If $\pi\colon L \to L$ is an orthomorphism, then*

$$|\pi(u)| = \big|\pi(|u|)\big| = |\pi|(|u|) \quad \textit{for all } u \in L.$$

Proof. The desired identity follows from the relations

$$\begin{aligned}|\pi|(|u|) \geqslant |\pi(u)| &= |\pi(u^+)-\pi(u^-)| = |\pi(u^+)|+|\pi(u^-)| \\ &= \pi(u^+)\vee[-\pi(u^+)]+\pi(u^-)\vee[-\pi(u^-)] = (\pi\vee-\pi)(u^+)+ \\ &\quad +(\pi\vee-\pi)(u^-) = |\pi|(u^+)+|\pi|(u^-) = |\pi|(|u|).\end{aligned}$$

The next result describes an important property of the domain of an orthomorphism (see [18], Theorem 1, p. 195).

THEOREM 1.6. *If two orthomorphisms agree on a set, then they agree on the band generated by that set.*

Proof. Let π_1 and π_2 be two orthomorphisms such that $\pi_1(u) = \pi_2(u)$ for each $u \in D$. We have to show that $\pi = \pi_1 - \pi_2 = 0$ on the band generated by D. Since, by Theorem 1.2, π is order continuous, it is enough to establish that $\pi = 0$ on the ideal generated by D.

To this end, let u be in the ideal generated by D. Then there exist $u_1, \ldots, u_n \in D$ and positive scalars $\lambda_1, \ldots, \lambda_n$ such that

$$|u| \leqslant \sum_{i=1}^{n} \lambda_i |u_i|.$$

Now by Lemma 1.1 we have

$$0 \leqslant |\pi(u)| = |\pi|(|u|) \leqslant \sum_{i=1}^{n} \lambda_i |\pi|(|u_i|) = \sum_{i=1}^{n} \lambda_i |\pi(u_i)| = 0,$$

so that $\pi(u) = 0$, and the proof is completed.

An immediate consequence of the preceding result is that the set where two orthomorphisms agree is always a band. In particular, the kernel of an orthomorphism is a band (since it is the set where the orthomorphism and the zero orthomorphism agree). Remarkably, every orthomorphism π satisfies $K_\pi = (R_\pi)^d$, where K_π is its kernel and R_π its range. The identity that the kernel is the disjoint complement of the range was shown in [5] and [8] via the representation theorems of Riesz spaces.

THEOREM 1.7. *If π is an orthomorphism on an Archimedean Riesz space, then $K_\pi = (R_\pi)^d$.*

Proof. If $u \in (R_\pi)^d$, then $u \perp \pi(v)$ for all $v \in L$, and so $\pi(u) \perp \pi(v)$ for each $v \in L$. In particular, $\pi(u) \perp \pi(u)$. That is, $\pi(u) = 0$ and, therefore, $u \in K_\pi$. Hence $(R_\pi)^d \subseteq K_\pi$.

Now let $u \in L$. Since L is Archimedean and K_π is an ideal, $K_\pi \oplus K_\pi^d$ is an order dense ideal ([10], Theorem 22.3, p. 114). Thus, there is a net $\{u_a + v_a\} \subseteq K_\pi \oplus K_\pi^d$ such that

$$u_a + v_a \xrightarrow{(o)} u.$$

By order continuity of π, we have $\pi(v_a) \xrightarrow{(o)} \pi(u)$. But, by Theorem 1.1, $\pi(v_a) \in K_\pi^d$ for each a, and so $\pi(u) \in K_\pi^d$. That is, $R_\pi \subseteq K_\pi^d$. Therefore, since K_π is a band, $K_\pi = K_\pi^{dd} \subseteq (R_\pi)^d$. Thus $K_\pi = (R_\pi)^d$.

Birkhoff and Pierce [7] showed that every Archimedean f-algebra is necessarily commutative. Zaanen using the formula $K_\pi = (R_\pi)^d$ was able to present a simple and elegant proof of this result ([18], Theorem 2, p. 196). He also pointed out, however, that this proof cannot be called "elementary" since it rests upon the identity $K_\pi = (R_\pi)^d$ whose proof was based upon the representation theorems of Archimedean Riesz spaces. By the above, we see that Zaanen's proof of "Every Archimedean f-algebra is commutative." can be carried through without representation theorems. We note also that Zaanen's arguments require only the conclusion of Theorem 1.6 (see [18], p. 196).

For our next discussion we shall need one result from [18]. Let L be an Archimedean f-algebra. Then for each $u \in L$ the (multiplication) operator $\pi_u(v) = uv$ for $v \in L$ is an orthomorphism on L. Conversely, every orthomorphism on an Archimedean f-algebra with unit is a multiplication operator ([18], Theorem 3, p. 196). A rephrasement of this result is the following

THEOREM 1.8. *Let L be an Archimedean f-algebra with unit. Then $u \mapsto \pi_u$ is a Riesz isomorphism from L onto* $\mathrm{Orth}(L)$, *i.e.*, $\mathrm{Orth}(L) = L$. *In particular*, $\mathrm{Orth}(\mathrm{Orth}(L)) = \mathrm{Orth}(L)$.

2. Extending orthomorphisms. In this section we shall deal with extensions of orthomorphisms and we shall need the concept of the universal completion of an Archimedean Riesz space. A Riesz space L is called *laterally complete* if every disjoint subset of L^+ has a supremum. If L is Archimedean, then there exists a unique (up to a Riesz isomorphism) universally complete (i.e., laterally and Dedekind complete) Riesz space L^u such that L is Riesz isomorphic to an order dense Riesz subspace of L^u. With appropriate identifications we have the Riesz subspace inclusions $L \subseteq L^\delta \subseteq L^u$ with L order dense in L^u. The Riesz space L^u is of the form $C^\infty(\Omega)$ for some Hausdorff, extremally disconnected, compact topological space Ω. If L has a weak order unit e, then the embedding of L into $C^\infty(\Omega)$ can be taken so that e corresponds to the constant function one on Ω; for details see [10], Section 50. It is important to observe that $C^\infty(\Omega)$, with Ω extremally disconnected, under the pointwise multiplication is an Archimedean f-algebra with unit element being the constant function one. Therefore, the universal completion L^u of an Archimedean Riesz space L is an f-algebra with unit.

Now consider an Archimedean Riesz space L and a positive orthomorphism π on L. By the discussion of the preceding section, π extends uniquely to an orthomorphism π^* on L^δ. By Theorem 1.2, π^* is a normal Riesz homomorphism on L^δ, and hence (by [3], Theorem 23.16, p. 172) π^* extends uniquely to a normal Riesz homomorphism from L^u into L^u, which we denote again by π^*. The extension π^* satisfies

$$\pi^*(u) = \sup\{\pi(v)\colon v \in L \text{ and } v \leqslant u\} \quad \text{for each } u \ (0 \leqslant u \in L^u).$$

It is a routine matter to show that π^* is likewise a positive orthomorphism on L^u. Consequently, every orthomorphism π on L extends uniquely to an orthomorphism π^* on L^u. Clearly, $\pi \mapsto \pi^*$ is a Riesz isomorphism from $\mathrm{Orth}(L)$ into $\mathrm{Orth}(L^u)$. If we identify $\mathrm{Orth}(L)$ with its image in $\mathrm{Orth}(L^u)$ under $\pi \mapsto \pi^*$, then we see that $\mathrm{Orth}(L)$ consists of all orthomorphisms on L^u that leave L invariant. In particular, since L^u is an Archimedean f-algebra, every orthomorphism on L is a "multiplication" operator; that is, if $\pi \in \mathrm{Orth}(L)$, then there exists some $u \in L^u$ such that $\pi(v) = uv$ for all $v \in L$.

Rephrasing the above discussion we have the following result:

THEOREM 2.1. *If L is an Archimedean Riesz space, then*

$$\mathrm{Orth}(L) = \{\pi \in \mathrm{Orth}(L^u)\colon \pi(L) \subseteq L\}.$$

The preceding theorem cannot be considered, of course, as "elementary". However, it is a very powerful theorem. Next we shall derive most of the results of [18] from Theorem 2.1.

Let L be an Archimedean Riesz space and let $e > 0$. Then there exists at most one product on L that makes L an f-algebra having e as its unit element. Indeed, if two products $\cdot$ and $*$ make L an f-algebra, with e as unit for both products, then for each fixed $v \in L$ the orthomorphism $\pi_v(u) = v\cdot u - v*u$ satisfies $\pi_v(e) = 0$. Since in this case e must be a weak order unit ($e \wedge w = 0$ implies $w = (e\cdot w)\wedge w = 0$), it follows from Theorem 1.6 that $\pi_v = 0$ on L, and so $v\cdot u = v*u$ for each $v, u \in L$.

THEOREM 2.2. *Let L be an Archimedean f-algebra with a unit e. Then there exists a unique product on L^u under which L^u is an f-algebra having the same unit e and containing L as an f-subalgebra.*

Proof. Embed L (order densely) in L^u in such a way that e corresponds to the constant function one of $C^\infty(\Omega) = L^u$. Denote by $\cdot$ the product operation on L, and by $*$ the pointwise multiplication on $C^\infty(\Omega)$.

For each fixed $u \in L$, the operator $\pi_u(v) = u\cdot v$ $(v \in L)$ is an orthomorphism on L, and hence, by Theorem 2.1, π_u extends to an orthomorphism on L^u. By Theorem 1.8 there exists some $w \in L^u$ such that $\pi_u(v) = w*v$ for all $v \in L^u$. In particular, $u = u\cdot e = \pi_u(e) = w*e = w$. Thus $u\cdot v = u*v$ for each $u, v \in L$, and so $*$ extends $\cdot$ to L^u. The uniqueness of $*$ follows from the remarks preceding the theorem.

It is worth observing that if $L = C(X)$, then although the extremally disconnected space Ω (where $L^u = C^\infty(\Omega)$) may have no direct relation with X, the preceding theorem shows that $C(X)$ can be embedded in $C^\infty(\Omega)$ in such a way that the two pointwise multiplications agree.

Examples of orthomorphisms. All examples below have appeared in [18]. Next, we shall show how to obtain them from the preceding discussion.

1. *Let $L = c_0(X)$ for some non-empty set X. Then $u \mapsto \pi_u$, where $\pi_u(v) = uv$ for each $v \in c_0(X)$, is a Riesz isomorphism from $l_\infty(X)$ onto* Orth(L), *and so* Orth$(L) = l_\infty(X)$.

The difficult part is to show that the mapping is onto. To see this let $\pi \in \mathrm{Orth}(L)$ and note that the universal completion of $c_0(X)$ is $\boldsymbol{R}^X$. By Theorem 2.1, π extends to an orthomorphism on $\boldsymbol{R}^X$, and so there exists some $u \in \boldsymbol{R}^X$ such that $\pi(v) = uv$ (pointwise product) for each $v \in L$. An easy argument now shows that $u \in l_\infty(X)$, and hence $\pi = \pi_u$.

2. *Let X be a locally compact Hausdorff topological space and let $L = C_c(X)$, the Riesz space of all continuous real-valued functions on X with compact support. For each $u \in C(X)$, the operator $\pi_u(v) = uv$ for $v \in C_c(X)$*

is an orthomorphism, and $u \mapsto \pi_u$ *is a Riesz isomorphism from* $C(X)$ *onto* $\mathrm{Orth}(L)$, *and so* $\mathrm{Orth}\big(C_c(X)\big) = C(X)$.

Again the difficult part is to show that the above mapping is onto. To this end, let $\pi \in \mathrm{Orth}(L)$ and observe that $C(X)$ is an Archimedean f-algebra with unit. Embed $C(X)$ in its universal completion according to Theorem 2.2. Since L is order dense in $C(X)$, the universal completion of $C(X)$ equals L^u ([3], Theorem 23.21, p. 175); thus $L \subseteq C(X) \subseteq L^u$. By Theorem 2.1, π can be considered as an orthomorphism on L^u. Therefore, there exists some $u \in L^u$ such that $\pi(v) = uv$ for each $v \in L$; note that $u = \pi(e)$.

Next observe that for every open set V, the set

$$\{f \in C_c(X) : f = 0 \text{ on } V\}$$

is a band of L. Thus, by Theorem 1.1, if $f \in C_c(X)$ vanishes on some open set V, then $\pi(f) = 0$ on V. In particular, if $f, g \in C_c(X)$ satisfy $f = g$ on V, then they also satisfy $\pi(f) = \pi(g)$ on V. Now, for each open set V with compact closure, choose $f_V \in C_c(X)$ such that $0 \leqslant f \leqslant 1$ and $f_V = 1$ on V. Then $f_V(x) \uparrow 1$ for each $x \in X$, and so $\pi(f_V) \uparrow \pi(e) = u$. On the other hand, in view of $\pi(f_V) = \pi(f_W)$ on $V \cap W$, it follows that $\{\pi(f_V)\}$ converges pointwise to some continuous function. This means that $u \in C(X)$, and so $\pi = \pi_u$.

3. *Let* (X, Σ, μ) *be a* σ*-finite measure space and let* $L = L_p(\mu)$ *with* $0 < p < \infty$. *Then* $u \mapsto \pi_u$, *where* $\pi_u(v) = uv$ *for* $v \in L$, *is a Riesz isomorphism from* $L_\infty(\mu)$ *onto* $\mathrm{Orth}(L)$, *and so* $\mathrm{Orth}\big(L_p(\mu)\big) = L_\infty(\mu)$.

We need only to prove that the mapping is onto. To see this, let $\pi \in \mathrm{Orth}(L)$. Note that $L^u = \mathcal{M}$, the Riesz space of all equivalence classes of μ-measurable functions. By Theorem 2.1, π can be considered as an orthomorphism on $\mathcal{M}$, and so there exists some $u \in \mathcal{M}$ such that $\pi(v) = uv$ for each $v \in L$. From standard arguments it now follows easily that $u \in L_\infty(\mu)$, and so $\pi = \pi_u$.

Note. For related work on orthomorphisms see [2], [4], [13], [15], and [17].

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