COLLOQUIUM MATHEMATICUM

VOL. LIV 1987 FASC. 1

A COUNTEREXAMPLE ON GENERALIZED CONVOLUTIONS

BY

K. URBANIK (WROCŁAW)

Notation and preliminaries. The aim of this note is to solve a problem concerning the regularity of quasi-regular generalized convolutions posed by Kucharczak in [1], P 826. Let us recall some definitions. We denote by $\mathscr{P}$ the set of all probability measures defined on Borel subsets of the positive half-line R^+. The set $\mathscr{P}$ is endowed with the topology of weak convergence. For $\mu \in \mathscr{P}$ and $a > 0$ we define the map T_a by setting $(T_a \mu)(E) = \mu(a^{-1} E)$ for all Borel subsets E of R^+. By δ_c we denote the probability measure concentrated at the point c. For $\mu, \nu \in \mathscr{P}$ we denote by $\mu\nu$ the probability distribution of the product XY of two independent random variables X and Y with probability distributions μ and ν, respectively. The operation $\mu\nu$ is a commutative semigroup operation with the properties

$$T_a \mu = \delta_a \mu \quad (a > 0,\ \mu \in \mathscr{P}), \tag{1}$$

$$\big(c\mu + (1-c)\nu\big)\lambda = c(\mu\lambda) + (1-c)(\nu\lambda) \quad (0 \leqslant c \leqslant 1,\ \mu, \nu, \lambda \in \mathscr{P}). \tag{2}$$

Moreover, we have the following simple relations:

PROPOSITION 1. *If $\mu_n \to \mu$, then $\mu_n \nu \to \mu\nu$.*

PROPOSITION 2. *If $\nu \neq \delta_0$ and $\mu_n \nu \to \lambda$, then the sequence μ_n is conditionally compact and each its limit point μ fulfils the equation $\mu\nu = \lambda$.*

A measure ν from $\mathscr{P}$ is said to be *cancellable* if the equation $\nu\mu = \nu\lambda$ yields $\mu = \lambda$. Propositions 1 and 2 imply the following statement:

PROPOSITION 3. *For cancellable measures ν the relation $\mu_n \nu \to \lambda$ holds if and only if $\mu_n \to \mu$ and $\mu\nu = \lambda$.*

A continuous in each variable separately commutative and associative $\mathscr{P}$-valued binary operation $\circ$ on $\mathscr{P}$ is called a *generalized convolution* if it is distributive with respect to convex combinations and maps T_a ($a > 0$) with δ_0 as the unit element. Moreover, the key axiom postulates the existence of constants c_n and a measure $\gamma \in \mathscr{P}$ other than δ_0 such that

$$T_{c_n} \delta_1^{\circ n} \to \gamma. \tag{3}$$

The power $\delta_1^{\circ n}$ is taken here in the sense of the operation $\circ$. The set $\mathscr{P}$ with the operation $\circ$ and all operations of convex combinations is called a *generalized convolution algebra* and is denoted by $(\mathscr{P}, \circ)$. For basic properties of generalized convolution algebras we refer to [2]. In particular, generalized convolution algebras admitting a non-constant continuous homomorphism into the algebra of real numbers with the operations of multiplication and convex combinations are called *regular*. The class of regular algebras coincides with the class of all algebras admitting an analogue of characteristic functions ([2], Theorem 6). A generalized convolution algebra is said to be *quasi-regular* if in condition (3) we assume in addition that the sequence c_n tends to 0. This notion has been introduced by Kucharczak in [1]. It has been proved in [2] (Theorem 4) that every regular generalized convolution algebra is quasi-regular. Kucharczak asked in [1] (P 826) whether the converse implication is true. We shall answer this question in the negative constructing a family of quasi-regular and not regular generalized convolution algebras.

We begin with the following simple lemma:

LEMMA. *Suppose that a generalized convolution algebra $(\mathscr{P}, \circ)$ is regular and*

$$\delta_x \circ \delta_1 = x\delta_1 \circ \delta_1 + (1-x)\delta_1 \qquad (0 \leqslant x \leqslant 1). \tag{4}$$

Then the measure $\delta_1 \circ \delta_1$ is absolutely continuous.

Proof. It has been proved in [2] (Theorem 6) that each regular generalized convolution algebra admits a *characteristic function*, i.e. a continuous isomorphism from $(\mathscr{P}, \circ)$ into the algebra of all real-valued bounded continuous functions on R^+ with the topology of uniform convergence on every compact subset of R^+ and with the operations of pointwise multiplication and convex combinations. Moreover, the characteristic function is an integral transform

$$\hat{\mu}(t) = \int_0^\infty \Omega(tu)\,\mu(du) \qquad (t \in R^+). \tag{5}$$

Here the kernel Ω is non-constant and continuous on R^+, $\Omega(0) = 1$, and $|\Omega(t)| \leqslant 1$ for $t \in R^+$. Equation (4) can be written in terms of characteristic functions as follows:

$$\Omega(xt)\,\Omega(t) = x\Omega^2(t) + (1-x)\,\Omega(t) \qquad (0 \leqslant x \leqslant 1,\ t \in R^+).$$

Put $A = \{t\colon t \in R^+,\ \Omega(t) \neq 0\}$. Since $0 \in A$, the set A is non-void and $a = \sup A > 0$. Moreover, the last equation yields

$$\Omega(xt) = x\Omega(t) + 1 - x \qquad (0 \leqslant x \leqslant 1,\ t \in A). \tag{6}$$

Let $a_n \in A$ and $a_n \to a$. Given $t < a$ we put $x_n = t/a_n$. Then $0 < x_n < 1$ for sufficiently large n and, by (6),

$$\Omega(t) = \Omega(x_n a_n) = x_n \Omega(a_n) + 1 - x_n \tag{7}$$

for sufficiently large n. Suppose that $a = \infty$. Then $x_n \to 0$ and, by (7), $\Omega(t) = 1$ for all $t \in R^+$, which yields a contradiction. Thus $a < \infty$ and, by the continuity of Ω, $\Omega(t) = 0$ for $t \geqslant a$. Further, taking into account (7) we infer that $\Omega(t) = 1 - t/a$ for $t < a$. Consequently,

$$\Omega(t) = \max(1 - t/a, 0)$$

for a certain $a > 0$. By standard computations we get the formula

$$(\delta_1 \circ \delta_1)^\wedge(t) = \Omega^2(t) = 2 \int_1^\infty \Omega(tu) u^{-3} du$$

which, by (5), yields the assertion of the Lemma.

Counterexample. For any pair μ, $\nu \in P$ we shall denote by $\mu \Box \nu$ the probability distribution of $\max(X, Y)$, where the random variables X and Y have the probability distributions μ and ν, respectively. It is clear that $(\mathscr{P}, \Box)$ is a generalized convolution algebra. Let p be an arbitrary real number satisfying the condition $0 < p < 1$. Put

$$\lambda(E) = (1-p)\delta_1(E) + p \int_{E \cap [1, \infty)} u^{-2} du$$

for all Borel subsets E of R^+. By standard computations we have

$$(\lambda\delta_0) \Box (\lambda\delta_0) = \lambda\delta_0, \tag{8}$$

$$(\lambda\delta_a) \Box (\lambda\delta_b) = \lambda\left[\left(1 - p\frac{a}{b}\right)\delta_b + p\frac{a}{b} T_b \sigma\right] \quad (a \leqslant b,\ b > 0), \tag{9}$$

where σ is absolutely continuous on R^+ with the density g given by the formula

$$g(u) = \begin{cases} \dfrac{2p}{(2p-1)u^3} - \dfrac{1}{(2p-1)u^{1+1/(1-p)}} & (p \neq \frac{1}{2},\ u \geqslant 1), \\ \dfrac{1 + 2\log u}{u^3} & (p = \frac{1}{2},\ u \geqslant 1), \\ 0 & \text{otherwise.} \end{cases}$$

Taking into account (1), (2) and the distributivity of $\Box$ with respect to convex combinations we infer, by Proposition 1, that for every pair μ, $\nu \in \mathscr{P}$ there exists a measure $\sigma(\mu, \nu)$ satisfying the equation

$$(\lambda\mu) \Box (\lambda\nu) = \lambda\sigma(\mu, \nu). \tag{10}$$

In particular, for every n there exists a measure $\gamma_n \in \mathscr{P}$ such that

$$T_{1/n}\lambda^{\square n} = (\lambda\delta_{1/n})^{\square n} = \lambda\gamma_n. \tag{11}$$

Observe that

$$T_{1/n}\lambda^{\square n}([0, x)) = (1-p/nx)^n \quad \text{if } x \geqslant 1/n,$$

which yields the convergence

$$T_{1/n}\lambda^{\square n} \to \varrho, \tag{12}$$

where $\varrho([0, x)) = \exp(-p/x)$ $(x \in R^+)$. By (11) and Proposition 2 the sequence γ_n is conditionally compact and each its limit point γ fulfils the equation

$$\varrho = \lambda\gamma. \tag{13}$$

Now we are ready to prove that the measure λ is cancellable. Suppose that $\lambda\mu = \lambda\nu$. Then, by (13), $\varrho\mu = \varrho\nu$. Since

$$(\varrho\beta)([0, x)) = \int_0^\infty \exp(-pu/x)\,\beta(du) \qquad (x \in R^+,\ \beta \in \mathscr{P}),$$

from the last equation we conclude that the Laplace transforms of μ and ν are identical. Thus $\mu = \nu$, which shows that the measure λ is cancellable. Hence the map $(\mu, \nu) \to \sigma(\mu, \nu)$ in (10) is well defined. Let us introduce the notation $\mu \circ \nu = \sigma(\mu, \nu)$. We shall prove that $\circ$ is a generalized convolution. Since λ is cancellable, we infer, by Proposition 3, that the operation $\circ$ is continuous in each variable separately. By (1), (2) and (10), $\circ$ is a commutative semigroup operation distributive with respect to operations T_a $(a > 0)$ and convex combinations. Moreover, by (8) and (9), $\delta_0 \circ \delta_b = \delta_b$ $(b \in R^+)$, which yields $\delta_0 \circ \mu = \mu$ for every $\mu \in \mathscr{P}$. Further, by (9) we have

$$\delta_a \circ \delta_b = \left(1 - p\frac{a}{b}\right)\delta_b + p\frac{a}{b}T_b\sigma \qquad (a \leqslant b,\ b > 0). \tag{14}$$

Finally, by (11), $T_{1/n}\delta_1^{\circ n} = \gamma_n$, which by (12) and Proposition 3 yields the convergence

$$T_{1/n}\delta_1^{\circ n} \to \gamma, \tag{15}$$

where the measure γ fulfils equation (13). Since $\varrho \neq \delta_0$, we have also $\gamma \neq \delta_0$, which shows that the operation $\circ$ fulfils condition (3). This completes the proof that $\circ$ is a generalized convolution. By (15) the generalized convolution algebra $(\mathscr{P}, \circ)$ is quasi-regular. It remains to prove that $(\mathscr{P}, \circ)$ is not regular. Observe that, by (14),

$$\delta_1 \circ \delta_1 = (1-p)\,\delta_1 + p\sigma \tag{16}$$

and

$$\delta_x \circ \delta_1 = (1-px)\,\delta_1 + px\sigma \qquad (0 \leqslant x \leqslant 1),$$

whence the equation

$$\delta_x \circ \delta_1 = x\delta_1 \circ \delta_1 + (1-x)\delta_1 \qquad (0 \leqslant x \leqslant 1)$$

follows. By (16), $(\delta_1 \circ \delta_1)(\{1\}) = 1-p > 0$, which shows that the measure $\delta_1 \circ \delta_1$ is not absolutely continuous. Applying the Lemma we infer that the generalized convolution algebra $(\mathscr{P}, \circ)$ is not regular.

REFERENCES

[1] J. Kucharczak, *A characterization of α-convolutions*, Colloq. Math. 27 (1973), pp. 141–147.

[2] K. Urbanik, *Generalized convolutions*, Studia Math. 23 (1964), pp. 217–245.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY

Reçu par la Rédaction le 10.2.1984