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CERTAIN SUFFICIENT CONDITIONS TO BE A COMPLEX PROJECTIVE SPACE

BY

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1. Introduction. Let (M, g) be an n-dimensional connected Riemannian manifold with metric tensor g. By Δ ($\Delta = g^{ij}\nabla_i\nabla_j$) we mean the Laplacian acting on the space of C^{∞}-functions f on (M, g), where ∇ is the operator of covariant derivation with respect to the connection induced by g, and we assume that the indices $h, i, j, \ldots$ run over the range $\{1, 2, \ldots, n\}$. Recently, in [1], [3], [5], [7], the eigenvalues λ of the Laplacian ($\Delta f + \lambda f = 0$) were calculated by an interesting method and the relations between λ and the curvature on (M^2, g) and the Einstein manifold (M^n, g), $n \geqslant 3$, were studied.

Let (M, J, g) be a real n-dimensional ($n = 2m$) Kähler manifold with complex structure tensor J and Kähler metric tensor g.

The main purpose of this paper is to show the following result which will be proved by a method similar to that used in [1].

THEOREM 1. *Let (M, J, g) be a closed and connected Kähler manifold of complex dimension m. If the equation $\Delta f + \lambda f = 0$ admits a non-zero solution and $(2R_{ij} - \lambda g_{ij})\bar{f}^i\bar{f}^j$ is positive semi-definite, then each eigenvalue satisfies*

$$\lambda \geqslant 4(m+1)\varkappa_0,$$

where $\bar{f}_i = J_i^{\ r} f_r$ and $\varkappa_0$ is the minimum of all sectional curvatures. The eqality holds iff (M, J, g) is holomorphically isometric to the complex m-dimensional projective space (CP^m, J, g_0) with the Fubini-Study metric g_0 of constant holomorphic sectional curvature $4\varkappa_0$.

Remark. The first eigenvalue of the Laplacian on a complex projective space (CP^m, J, g_0) with the Fubini-Study metric of constant holomorphic sectional curvature k is $(m+1)k$.

The second purpose of the paper is to obtain the following theorem by means of the integral formula on a K-conformal Killing tensor.

THEOREM 2. *Let (M, J, g) be a compact Kähler-Einstein manifold. If (M, J, g) admits a Killing vector field u^i such that $\nabla_i \bar{u}^i$ $(= J^{ij}\nabla_i u_j)$ is not constant and $u^{kj}u^{ih}P_{kjih} \geqslant 0$, then (M, J, g) is holomorphically isometric to (CP^m, J, g_0).*

2. Preliminaries. Let (M, g) be an n-dimensional Riemannian manifold (connected and C^∞) with metric tensor g. In (M, g), if we put $f_{ji} = \nabla_j \nabla_i f$, where f is a C^∞-function, then f_{ji} denote the components of the Hessian Hess(f). Hence $\Delta f = g^{ji} f_{ji}$. If f satisfies $\Delta f + \lambda f = 0$, then it is called the *eigenfunction corresponding to the eigenvalue* λ. By $R_{kji}{}^h$, R_{ji}, and R we denote the Riemannian curvature tensor, the Ricci tensor, and the scalar curvature, respectively.

To prove Theorem 1, we need the following lemmas obtained by Simon [5], [6]:

LEMMA 2.1. *Let f be a C^∞-function. Then f satisfies the equation*

$$\tfrac{1}{2}\Delta(f_{ji} f^{ji}) = 2\sum_{i<j} \varkappa_{ij}(\sigma_i - \sigma_j)^2 + f^{ji} \nabla_j \nabla_i (\Delta f) + $$
$$+ \nabla_k f_{ji} \nabla^k f^{ji} + f^{ji} f^k (2\nabla_i R_{jk} - \nabla_k R_{ji}),$$

where $\sigma_1, \ldots, \sigma_n$ are the eigenvalues of the Hessian (with the corresponding orthonormal eigenvectors $E_1, \ldots, E_n$) and $\varkappa_{ij}$ is the sectional curvature of the plane $\{E_i, E_j\}_{i \neq j}$.

LEMMA 2.2. *Let (M, g) be closed $(n \geqslant 2)$. Let f and h be C^∞-functions. Then*

$$\int f_{ji} h^{ji}\, do - \int \Delta f \Delta h\, do + \int R^{ji} f_j h_i\, do = 0,$$

where do means the volume element of (M, g).

LEMMA 2.3. *Let (M, g) be closed $(n \geqslant 2)$. Then each eigenfunction f corresponding to the eigenvalue λ satisfies the equation*

$$(n-1)\lambda \int \|df\|^2\, do = n \int R^{ji} f_j f_i\, do + \int \sum_{i<j} (\sigma_i - \sigma_j)^2\, do.$$

Next, we recall the definition of a Kähler manifold (M, J, g) and identities which are necessary in the sequel. In (M, J, g), the following identities hold:

$$J_i{}^r J_r{}^j = -\delta_i{}^j, \quad g_{rs} J_i{}^r J_j{}^s = g_{ij},$$
$$\nabla_k J_i{}^j = 0, \quad J_{ij} = -J_{ji}, \quad R_i{}^r J_r{}^j = -\tfrac{1}{2} R_{rsi}{}^j J^{rs},$$

where $J_{ij} = J_i{}^r g_{rj}$.

The holomorphically projective curvature tensor $P_{kji}{}^h$ is given by

$$P_{kji}{}^h = R_{kji}{}^h + \frac{1}{n+2}(R_{ki}\delta_j{}^h - R_{ji}\delta_k{}^h + S_{ki} J_j{}^h - S_{ji} J_k{}^h + 2S_{kj} J_i{}^h),$$

where $S_{ji} = J_j{}^r R_{ri}$. A necessary and sufficient condition for $P_{kji}{}^h = 0$ is that the manifold is a space of constant holomorphic sectional curvature c, i.e., a space whose curvature tensor $R_{kji}{}^h$ takes the form

$$R_{kji}{}^h = \frac{c}{4}(g_{ji}\delta_k{}^h - g_{ki}\delta_j{}^h + J_{ji} J_k{}^h - J_{ki} J_j{}^h - 2J_{kj} J_i{}^h).$$

The following theorem, that was announced by Obata [4] and proved by Tanno [9], plays very important roles in the proofs of our theorems.

THEOREM A. *Let (M, J, g) be a complete connected Kähler manifold of complex dimension m. In order for (M, J, g) to admit a non-constant function f satisfying the following system of differential equations of order three for some positive constant c:*

$$4\nabla_k \nabla_j f_i + c(2f_k g_{ji} + f_j g_{ki} + f_i g_{kj} - \bar{f}_j J_{ki} - \bar{f}_i J_{kj}) = 0, \tag{2.1}$$

where $\bar{f}_j = J_j{}^r f_r$, it is necessary and sufficient that (M, J, g) is holomorphically isometric to the complex m-dimensional projective space (CP^m, J, g_0) with the Fubini-Study metric of constant holomorphic sectional curvature c.

LEMMA 2.4 ([12], p. 88). *If, in a compact Kähler manifold, the form $(2R_{ij} - \lambda g_{ij})\bar{f}^i \bar{f}^j$ is positive semi-definite, then $\bar{f}^i = J^{ir} f_r$ is a Killing vector for a solution of the equation $\Delta f + \lambda f = 0$.*

3. Proof of Theorem 1. First we define $B(f)_{kji}$ as

$$B(f)_{kji} = \nabla_k \nabla_j f_i + \frac{\lambda}{2(n+2)}(2f_k g_{ji} + f_j g_{ki} + f_i g_{kj} - \bar{f}_j J_{ki} - \bar{f}_i J_{kj}), \tag{3.1}$$

whence

$$\|B(f)_{kji}\|^2 = \|\nabla_k \nabla_j f_i\|^2 - \frac{2}{n+2}\lambda^2 \|df\|^2. \tag{3.2}$$

On the other hand, by the assumption that $(2R_{ij} - \lambda g_{ij})\bar{f}^i \bar{f}^j$ is positive semi-definite, Lemma 2.4 implies that $\bar{f}^i$ is Killing. So we have

$$\nabla_k \nabla_j \bar{f}_i + \bar{f}^r R_{rkji} = 0. \tag{3.3}$$

Transvecting (3.3) with $J_s{}^k$, we obtain

$$J_k{}^r \nabla_r \nabla_j \bar{f}_i - f^r R_{rkji} = 0.$$

Applying ∇^j to the above equation, we get

$$J_k{}^r J_i{}^s (\nabla_r \nabla^j \nabla_j f_s + R_r{}^t f_{ts} - R_{jrst} f^{jt}) - f^{jr} R_{rkji} - f^r (\nabla_k R_{ri} - \nabla_r R_{ki}) = 0.$$

Moreover, contracting (3.3) with $J^i{}_s g^{kj}$, we obtain

$$\nabla^j \nabla_j f_s + f^j R_{js} = 0, \tag{3.4}$$

whence

$$f^{ki} J_k{}^r J_i{}^s (-\nabla_r (f^j R_{js}) + R_r{}^t f_{ts} - R_{jrst} f^{jt}) -$$
$$- f^{ki} f^{jr} R_{rkji} - f^{ki} f^r (\nabla_k R_{ri} - \nabla_r R_{ki}) = 0.$$

Consequently, we have

$$f^{ki} f^r (2\nabla_k R_{ri} - \nabla_r R_{ki}) = 0.$$

Therefore, by (3.2) and $\Delta f + \lambda f = 0$, the formula from Lemma 2.1 can be rewritten as follows:

$$(3.5)\qquad 0 = \int\left[2\sum_{i<j} \varkappa_{ij}(\sigma_i - \sigma_j)^2 - \lambda \|f_{ji}\|^2 + \|B(f)_{kji}\|^2 + \frac{2}{n+2}\lambda^2 \|df\|^2\right] do.$$

In this way, making use of (3.4) and the Ricci identity, we obtain

$$(3.6)\qquad R_{ji} f^j f^i = \tfrac{1}{2}\lambda \|df\|^2,$$

which together with Lemma 2.2 yields

$$(3.7)\qquad 0 = \int [\|f_{ji}\|^2 - \lambda \|df\|^2 + R_{ji} f^j f^i]\, do = \int [\|f_{ji}\|^2 - \tfrac{1}{2}\lambda \|df\|^2]\, do.$$

Similarly, putting (3.6) into the equation in Lemma 2.3, we get

$$(n-1)\lambda \int \|df\|^2 do = \frac{n}{2}\lambda \int \|df\|^2 do + \int \sum_{i<j} (\sigma_i - \sigma_j)^2 do,$$

whence

$$(3.8)\qquad (n-2)\lambda \int \|df\|^2 do = 2\int \sum_{i<j} (\sigma_i - \sigma_j)^2 do.$$

Putting (3.7) and (3.8) into (3.5), we obtain

$$\begin{aligned} 0 &= \int\left[2\sum_{i<j} \varkappa_{ij}(\sigma_i - \sigma_j)^2 + \|B(f)_{kji}\|^2 - \frac{n-2}{2(n+2)}\lambda^2 \|df\|^2\right] do \\ &= \int\left[\sum_{i<j}\left(2\varkappa_{ij} - \frac{\lambda}{n+2}\right)(\sigma_i - \sigma_j)^2 + \|B(f)_{kji}\|^2\right] do \\ &\geqslant \int\left[\sum_{i<j}\left(2\varkappa_0 - \frac{\lambda}{n+2}\right)(\sigma_i - \sigma_j)^2 + \|B(f)_{kji}\|^2\right] do, \end{aligned}$$

which implies $2\varkappa_0 - \lambda/(n+2) \leqslant 0$, i.e.,

$$\lambda \geqslant 2(n+2)\varkappa_0 = 4(m+1)\varkappa_0.$$

If the equality holds, we have

$$0 = B(f)_{kji} = \nabla_k \nabla_j f_i + \varkappa_0(2f_k g_{ji} + f_j g_{ki} + f_i g_{kj} - \tilde{f}_j J_{ki} - \tilde{f}_i J_{kj}).$$

Applying Theorem A, we see that (M, J, g) is holomorphically isometric to the complex projective space (CP^m, J, g_0). Thus, the proof of the theorem is completed.

4. Proof of Theorem 2. In this section, making use of the integral formula on a K-conformal Killing tensor which was introduced in [11] by

Yamaguchi, i.e., a *K-conformal Killing tensor* is a skew symmetric tensor field u_{ij} satisfying

$$\nabla_i u_{jk} + \nabla_j u_{ik} = 2\varrho_k g_{ij} - \varrho_i g_{jk} - \varrho_j g_{ik} + 3(\tilde{\varrho}_i J_{jk} + \tilde{\varrho}_j J_{ik}),$$

where

$$\varrho_i = \frac{1}{n+2} \nabla^r u_{ri} \quad \text{and} \quad \tilde{\varrho}_i = J_i{}^r \varrho_r,$$

we study the relation between the K-conformal Killing tensor u_{ij} and the holomorphically projective curvature tensor $P_{kji}{}^h$ and we prove Theorem 2.

In [11], the following theorem is proved:

THEOREM B. *In a compact Kähler manifold* (M, J, g), $\dim M = n = 2m$, *the following integral formula is valid for any skew symmetric tensor* u_{ij}:

$$\int [(\nabla^r \nabla_r u_{ij} - R_i{}^r u_{jr} - R^r{}_{ij}{}^s u_{rs} + n\varrho_{ij} + \varrho_{ji} - \\ - 3\tilde{\varrho}_r{}^r J_{ij} + 3\tilde{\varrho}_{ri} J_j{}^r) u^{ij} + \tfrac{1}{2} \|A_{ijk}\|^2]\, do = 0,$$

where

$$\varrho_{ij} = \frac{1}{n+2} \nabla_i \nabla^r u_{rj},$$

$$\tilde{\varrho}_{ij} = J_j{}^r \varrho_{ir} = \frac{1}{3(n+2)} J^{rs} (\nabla_i \nabla_j u_{rs} + \nabla_i \nabla_r u_{js}),$$

and

$$A_{ijk} = \nabla_i u_{jk} + \nabla_j u_{ik} - 2\varrho_k g_{ij} + \varrho_i g_{jk} + \varrho_j g_{ik} - 3(\tilde{\varrho}_i J_{jk} + \tilde{\varrho}_j J_{ik}).$$

Let us show the following

THEOREM 4.1. *Let* (M, J, g) *be a compact Kähler manifold with a parallel Ricci tensor. If* (M, J, g) *admits a Killing vector field* u^i *and* $u^{kj} u^{ih} P_{kjih} \geqslant 0$ *is satisfied, then* $\nabla_j u_i$ *is a closed K-conformal Killing tensor field.*

Proof. We now put

$$T_{ij} = \nabla^r \nabla_r u_{ij} - R_i{}^r u_{jr} - R^r{}_{ij}{}^s u_{rs} + n\varrho_{ij} + \varrho_{ji} - 3\tilde{\varrho}_r{}^r J_{ij} + 3\tilde{\varrho}_{ri} J_j{}^r.$$

Let us calculate $u^{ij} T_{ij}$. As u^i is a Killing vector field, we have

$$\nabla_i \nabla_j u_k + u^r R_{rijk} = 0,$$

which means that $\nabla_j u_k$ is closed. Putting $\nabla_i u_j = u_{ij}$, we get

$$\varrho_{ij} = -\frac{1}{n+2} u_i{}^r R_{rj},$$

$$\tilde{\varrho}_r{}^r J_{ij} = -\frac{1}{n+2} u_r{}^t R_{ts} J^{rs} J_{ij}, \qquad \tilde{\varrho}_{ri} J_j{}^r = -\frac{1}{n+2} u_r{}^t R_{ts} J_i{}^s J_j{}^r,$$

whence

$$(4.1)\qquad u^{ij}T_{ij}=\frac{3}{2}u^{ij}u^{rs}R_{ijrs}-u^{ij}R_i{}^r u_{jr}+(n-1)u^{ij}\varrho_{ij}-3u^{ij}(\tilde{\varrho}_r{}^r J_{ij}-\tilde{\varrho}_{ri}J_j{}^r)$$
$$=\frac{3}{2}\Big[u^{ij}u^{rs}R_{ijrs}+\frac{2}{n+2}(u^{ij}u^r{}_j R_{ir}+u^{ij}u^{rt}J_r{}^s J_{ij}R_{ts}+ +u^{ij}u^{rt}J_i{}^s J_{rj}R_{ts})\Big].$$

On the other hand, after a straightforward calculation we get

$$u^{kj}u^{ih}P_{kjih}=u^{kj}u^{ih}R_{kjih}+\frac{2}{n+2}(u^{kj}u^i{}_j R_{ki}+u^{kj}u^{ih}J_k{}^r J_{jh}R_{ri}+u^{kj}u^{ih}J_k{}^r J_{ih}R_{rj}).$$

Therefore, comparing (4.1) with this formula, we obtain

$$u^{ij}T_{ij}=\frac{3}{2}u^{kj}u^{ih}P_{kjih}.$$

We can see by Theorem B that u_{ij} is a closed K-conformal Killing tensor field.

As a corollary to Theorem 4.1, we can prove Theorem 2.

Proof of Theorem 2. According to Theorem 4.1, we see that $\nabla_j u_i$ is a closed K-conformal Killing tensor field. So, we get

$$2\nabla_i u_{jk}-\nabla_k u_{ji}=2\varrho_k g_{ij}-\varrho_i g_{jk}-\varrho_j g_{ik}+3(\tilde{\varrho}_i J_{jk}+\tilde{\varrho}_j J_{ik}).$$

Since the left-hand side is equal to $3\nabla_i u_{jk}+\nabla_i u_{kj}+\nabla_k u_{ij}$ and (M, J, g) is an Einstein manifold, we obtain

$$\nabla_i u_{jk}=-\frac{R}{n(n+2)}(u_k g_{ij}-u_j g_{ik}+2\tilde{u}_i J_{jk}+\tilde{u}_j J_{ik}+\tilde{u}_k J_{ji}).$$

If we put $f=\nabla_i \tilde{u}^i$, then it is easy to see that f is a solution of (2.1) because $\nabla_i f=-(2R/n)\tilde{u}_i$. Hence our proof is completed by Theorem A.

As a consequence of Theorem 1, we have

COROLLARY. *Let (M, J, g) be a closed and connected Kähler-Einstein manifold of complex dimension m. If the equation $\Delta f+(R/m)f=0$ admits a non-zero solution, then*

$$\varkappa_0\leqslant\frac{R}{4m(m+1)}.$$

The equality holds iff (M, J, g) is holomorphically isometric to the complex m-dimensional projective space (CP^m, J, g_0).

Remark. Using Theorem 2 we can give another proof of the above corollary. Namely, since the equation $\Delta f+(R/m)f=0$ admits a non-zero solution, $\tilde{f}^i$ $(=J^{ir} f_r)$ is a Killing vector field for which $\nabla_i \tilde{f}^i$ is not constant. Consequently, if $\tilde{f}^{kj} \tilde{f}^{ih} P_{kjih} \geqslant 0$, then we obtain the desired result by means of the properties of $P_{kji}{}^h$ and Lemma 2.3.

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