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A REMARK ON ENGELER'S FILTER-IMAGES

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1. In paper [1] Engeler has defined an operation, called filtered image, on relational structures. Let us recall it.

Definition 1. Let $\boldsymbol{A}$ be a structure with the universe A, B a set, and $\mathscr{D}$ a filter over A^B. The *image of* $\boldsymbol{A}$ *under the filter* $\mathscr{D}$ is the structure similar to $\boldsymbol{A}$, with the universe B, and relations defined as follows:

If S is an n-ary relation in $\boldsymbol{A}$ and R is the corresponding relation on B, then

$$(1)\qquad \langle b_1, \ldots, b_n\rangle \in R \text{ iff } \{p \in A^B\colon \langle p(b_1), \ldots, p(b_n)\rangle \in S\} \in \mathscr{D}.$$

Equality is understood as identity in both structures.

In this paper a slight modification of the above definition is proposed: *equality will be treated as a relation in structures* (*not in the logic*).

For such filter-images the following characterization holds:

A structure $\boldsymbol{B}$ is the *filter-image* of a structure $\boldsymbol{A}$ iff $\boldsymbol{B}$ is isomorphic to a substructure of a reduced direct power of $\boldsymbol{A}$.

Consequently, all theorems given by Engeler are consequences or reformulations of theorems on reduced direct powers from, e.g., [2] and [3].

2. The meaning of the word "isomorphism" in the above statement needs an explanation.

Definition 2. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two similar structures with universes A and B respectively, and $\mathscr{L}$ be the language of the same type. A subset I of $A\times B$ (a binary relation between elements of A and B) is called an *isomorphism* if

(i) $I\circ I^{-1} \supset \{\langle a, a\rangle\colon a\in A\}$ and $I^{-1}\circ I \supset \{\langle b, b\rangle\colon b\in B\}$;

(ii) for every atomic formula Φ of $\mathscr{L}$ with n variables, $\langle a_1, b_1\rangle, \ldots$ $\ldots, \langle a_n, b_n\rangle \in I$ implies

$$\boldsymbol{A} \models \Phi[a_1, \ldots, a_n] \text{ iff } \boldsymbol{B} \models \Phi[b_1, \ldots, b_n].$$

$\boldsymbol{A} \simeq \boldsymbol{B}$ will denote that $\boldsymbol{A}$ and $\boldsymbol{B}$ are isomorphic in the above sense.

It seems that isomorphism in this sense was considered elsewhere but we cannot give any reference. The following easily provable propositions explain its nature:

PROPOSITION 1. *If I is an isomorphism, then I^{-1} is an isomorphism. If I and J are isomorphisms and $I \circ J$ is defined, then $I \circ J$ is an isomorphism.*

PROPOSITION 2. *If* Th$(\boldsymbol{A})$ *contains the theory of equality realized as a relation $\sim_{\boldsymbol{A}}$, and $\sim_{\boldsymbol{B}}$ is the corresponding relation in $\boldsymbol{B}$, and $\boldsymbol{A} \simeq \boldsymbol{B}$, then $\sim_{\boldsymbol{B}}$ is the equality, and $\boldsymbol{A}/\sim_{\boldsymbol{A}}$ and $\boldsymbol{B}/\sim_{\boldsymbol{B}}$ are isomorphic in the usual sense.*

A theorem converse to Proposition 2 is trivially true.

3. There arises the question whether Engeler's construction is identical with that of ours. This is the case if the relation in filter-image of $\boldsymbol{A}$ corresponding to the identity in $\boldsymbol{A}$ is identity itself. A condition on filters, necessary and sufficient for that, is the following:

$$\text{if } b_1 \neq b_2 \in B, \text{ then } \{p \in A^B\colon\ p(b_1) \neq p(b_2)\} \in \mathscr{D}. \tag{2}$$

This condition is formulated in [1] in the equivalent form (condition (a), p. 108) for ultrafilters only, and main results of [1] are proved for ultrafilters fulfilling (2). Hence our results are essentially stronger than those of Engeler.

4. THEOREM 1. *If $\boldsymbol{B}$ is the image of $\boldsymbol{A}$ under filter $\mathscr{D}$, then $\boldsymbol{B} \simeq \boldsymbol{B}' \subseteq \boldsymbol{A}^K_{\mathscr{D}}$, where $K = A^B$.*

Proof. With each element $b \in B$ we can correlate the function $\bar{b}$ from K into A such that

$$\bar{b}(f) = f(b). \tag{3}$$

Classes of equivalence under $\mathscr{D}$ of such functions will form the universe of $\boldsymbol{B}'$. Obviously, $\boldsymbol{B}' \subseteq \boldsymbol{A}^K_{\mathscr{D}}$.

Let Φ be an atomic formula in $\mathscr{L}$. Then

$$\boldsymbol{B}' \models \Phi[\bar{b}_1/\mathscr{D}, \ldots] \text{ iff}$$

$$\boldsymbol{A}^K_{\mathscr{D}} \models \Phi[\bar{b}_1/D, \ldots] \text{ iff } \{f \in K\colon\ \boldsymbol{A} \models \Phi[\bar{b}_1(f), \ldots]\} \in \mathscr{D}.$$

Because of (3) we have

$$\boldsymbol{B}' \models \Phi[\bar{b}_1/\mathscr{D}, \ldots] \text{ iff } \{f \in K\colon\ \boldsymbol{A} \models \Phi[f(b_1), \ldots]\} \in \mathscr{D}.$$

Hence $\boldsymbol{B}' \simeq \boldsymbol{B}$.

THEOREM 2. *If $\boldsymbol{B} \subseteq \boldsymbol{A}^K_{\mathscr{D}}$, then $\boldsymbol{B}$ is isomorphic to a filter-image of $\boldsymbol{A}$.*

Proof. Let B' be the set of all $f \in A^K$ such that $f/\mathscr{D} \in B$. With each $i \in K$ we can associate a function $\bar{i}\colon B' \to A$ such that $\bar{i}(f) = f(i)$. Let $\bar{K} = \{\bar{i}\colon i \in K\}$. Thus $\bar{K} \subseteq A^{B'}$. Let $U \subseteq A^{B'}$. Then $U \in \overline{\mathscr{D}}$ iff there exists a $V \in \mathscr{D}$ such that $\{\bar{i}\colon i \in V\} \subseteq U$.

Let $\overline{\boldsymbol{B}}$ be the filtered image of $\boldsymbol{A}$ under $\overline{\mathscr{D}}$ with the universe B', and Φ an atomic formula of $\mathscr{L}$. Then

$$\overline{\boldsymbol{B}} \models \Phi[b_1, \ldots] \text{ iff}$$

$$\{f \in A^{B'}\colon \boldsymbol{A} \models \Phi[f(b_1), \ldots]\} \in \overline{\mathscr{D}} \text{ iff}$$

$$\{i \in K\colon \boldsymbol{A} \models \Phi[\bar{i}(b_1), \ldots]\} \in \mathscr{D} \text{ iff}$$

$$\{i \in K\colon \boldsymbol{A} \models \Phi[b_1(i), \ldots]\} \in \mathscr{D} \text{ iff}$$

$$\boldsymbol{A}^K_{\mathscr{D}} \models \Phi[b_1/\mathscr{D}, \ldots] \text{ iff } \boldsymbol{B} \models \Phi[b_1/\mathscr{D}, \ldots].$$

5. Using Theorems 1 and 2 we can give an answer to the following question: under what conditions a filter-image $\boldsymbol{B}$ of $\boldsymbol{A}$ is isomorphic to a limit reduced power of $\boldsymbol{A}$?

There are well known characterizations of those subsets of reduced powers which are limit reduced powers. Most pretty is perhaps the following one (see, e.g., [3]):

(4) $\boldsymbol{B} \subseteq \boldsymbol{A}^K_{\mathscr{D}}$ *is isomorphic to a limit reduced power of* A *iff for every function* F *on* A *there exists a function* G *on* B *such that* $\{i\colon F(b_1(i), \ldots, b_n(i)) = G(b_1, \ldots, b_n)(i)\} \in \mathscr{D}$.

Our reformulation provides an answer to the question.

Condition (4) has been investigated in [1] in some equivalent form.

REFERENCES

[1] E. Engeler, *On structures defined by mapping filters*, Mathematische Annalen 167 (1966), p. 105-112.

[2] H. J. Keisler, *Limit ultrapowers*, Transactions of the American Mathematical Society 107 (1963), p. 382-408.

[3] A. Wojciechowska, *Limit reduced powers*, Colloquium Mathematicum 20 (1969), p. 203-208.

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