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MONOTONICITY OF MULTIPLICATIVE FUNCTIONS

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We say that a multiplicative function f is *non-decreasing* on $A \subset N$ if, for $a, b \in A$, $a < b$ implies $f(a) \leqslant f(b)$. Erdös [1] has shown that if f is a multiplicative function non-decreasing on N, then with a suitable non-negative k one must have $f(n) = n^k$. Moser and Lambek [2] have given a new proof of this result and Pisot and Schoenberg [4] have proved that if a multiplicative function f is non-decreasing on the set of all integers composed of three fixed primes, then, on this set, $f(a) = a^k$ holds with a suitable $k \geqslant 0$. Examples of other sets with the same property were given in [3]. Now we formulate four properties of a subset A of N and consider the problem of characterization of those sets A which have one of them.

PROPERTY I. *If a positive and multiplicative function f is non-decreasing on A, then, for all a in A, $f(a) = a^k$ with a suitable k.*

PROPERTY II. *If a nowhere vanishing multiplicative function f is non-decreasing on $A \cap [m, \infty)$ with a certain m, then $f(a) = a^k$ for all a in A.*

PROPERTY III (1). *If a positive multiplicative function f is non-decreasing on A, then $f(n) = n^k$ holds for all n.*

PROPERTY IV. *If a nowhere vanishing multiplicative function f is non-decreasing on $A \cap [m, \infty)$ with a certain m, then $f(n) = n^k$ for all n.*

Note that one can replace here the word "non-decreasing" by "non-increasing" or "monotone" without changing the family of sets enjoying the corresponding property.

Properties I-IV are maintained after deleting finitely many numbers from A. Indeed, this is clear for properties II and IV, and if a positive and multiplicative function f is non-decreasing for sufficiently large elements of A, then $g(n) = n^k f(n)$ with a sufficiently large k is non-decreasing on A. Thus properties I and III are also maintained.

(1) Prof. Schinzel informed the author that property III was considered by Dr. Freud from Budapest.

Evidently, II implies I, and IV implies III. Prof. Schinzel observed that also III implies II. The converse implications are false.

Although we are not in a position to give complete solutions to the four problems, we can prove a rather general theorem which can be used to give many examples of sets satisfying one of properties I-IV.

THEOREM. *Let $A \subset N$ and assume that A satisfies the following conditions:*

(i) *A contains an infinite set of mutually prime integers.*

(ii) *There exists an integer $q \geqslant 2$ such that if $a_1, a_2, \ldots, a_q \in A$ and $(a_i, a_j) = 1$ holds for all $i \neq j$, then the product $a_1 a_2 \ldots a_q$ belongs to A.*

(iii) *The infinite product*

$$\prod_{i=1}^{\infty} \max\left\{\min \frac{b}{a},\ \min \frac{a}{b}\right\}, \tag{1}$$

where max *is taken over $a \in A$ such that $2^i \leqslant a < 2^{i+1}$, the first* min *over $b \in A$ such that $a < b$, $(a, b) = 1$, and the second one over $b \in A \cup \{1\}$ such that $b < a$, $(a, b) = 1$, converges provided the undefined terms are assumed to be* 1.

Then A has property I.

Proof. Let W be the value of product (1) and let γ be any number of A exceeding W. Denote by B the set of all integers greater than γW which are prime to γ and can be represented as products of $q-1$ members of A which are mutually prime. If $q = 2$, then, of course,

$$B = \{a \in A:\ a > \gamma W, (a, \gamma) = 1\}.$$

Now we are going to show that for each b in B there exist sequences $\{\varphi_n\} \subset A$ and $\{\psi_n\} \subset A$ such that for each $n \in N$

$$(b, \varphi_n) = 1, \tag{2}$$

$$b\gamma < \varphi_n < b^{n+1}, \tag{3}$$

$$b\varphi_n < \varphi_{n+1}, \tag{4}$$

$$(b, \psi_n) = 1, \tag{5}$$

$$b^{n+1} < \psi_{n+1} < b\psi_n, \tag{6}$$

$$b < \psi_1 < b\gamma. \tag{7}$$

For $n = 1, 2, 3, \ldots$, we put

$$\varphi_1 = \min_{l > b\gamma} l \quad \text{and} \quad \varphi_{n+1} = \min_{l > b\varphi_n} l,$$

where $l \in A$ and $(b, l) = 1$.

It is easy to verify that (2), (4) as well as the left inequality of (3) are satisfied.

We have

$$\frac{\varphi_1}{b\gamma}\prod_{i=2}^{n}\frac{\varphi_i}{b\varphi_{i-1}} = \frac{\varphi_n}{b^n\gamma}. \tag{8}$$

Since $b\gamma$ and $b\varphi_{i-1}$ lie in A, we can find, for each factor of the last product, the corresponding greater or equal term of (1) and these terms will be distinct because of $b \geqslant 2$. If for some n we had $\varphi_n \geqslant b^{1+n}$, then the value of (8) would exceed W and, consequently, product (1) would exceed W, a contradiction.

Let

$$\psi_1 = \max_{l<b\gamma} l \quad \text{and} \quad \psi_{n+1} = \max_{l<b\psi_n} l,$$

where $l \in A$ and $(b, l) = 1$.

Of course, if the sequence $\{\psi_n\}$ is well defined, then condition (5) and the right inequalities of (6) and (7) are satisfied. Additionally, we have $\psi_n < b^{n+1}$. The condition $b^n < \psi_n$ and the existence of $\{\psi_n\}$ can be proved by induction.

Clearly, $b\gamma \in A$. Hence, if ψ_1 does not exist, then $W \geqslant b\gamma$, contrary to the definition of γ. Similarly, we put $b < \psi_1$.

Moreover, if ψ_{n+1} did not exist, then in view of $b\psi_n \in A$ we would have

$$W \geqslant b\psi_n \geqslant b,$$

which is impossible.

We have also

$$\frac{b\gamma}{\psi_1}\prod_{i=2}^{n+1}\frac{b\psi_{i-1}}{\psi_i} = \frac{b^{n+1}\gamma}{\psi_{n+1}},$$

and if $\psi_{n+1} < b^{n+1}$, then the same argument as above would lead to $W < W$.

Now assume that f is a positive multiplicative function non-decreasing on A and let a, b belong to B. By (2) and (4) we get

$$f(a)f(\varphi_n) \leqslant f(\varphi_{n+1}).$$

Hence (3) implies

$$f(\gamma)f^n(a) \leqslant f(\varphi_n).$$

From (5) and (6) we obtain

$$f(\psi_{n+1}) \leqslant f(b)f(\psi_n),$$

and now (7) gives

$$f(\psi_n) \leqslant f(\gamma) f^n(b).$$

Since $a, b \geqslant 2$, there exist infinitely many triples (u, t, m) of positive integers satisfying

(9) $$a^u \leqslant m < b^{1+t} \quad \text{and} \quad b^t \leqslant m < a^{u+1}.$$

By (3), (6) and (9) we get $\varphi_{u-1} < m < \psi_{t+1}$, and thus

$$f(\gamma) f^{u-1}(a) \leqslant f(\gamma) f^{t+1}(b), \qquad f^{\log m/\log a - 2}(a) \leqslant f^{\log m/\log b + 2}(b),$$

$$f^{1/\log a - 2/\log m}(a) \leqslant f^{1/\log b + 2/\log m}(b).$$

As m may be arbitrarily large, this implies

$$f^{1/\log a}(a) \leqslant f^{1/\log b}(b),$$

and since a and b are arbitrary elements of B, we get

$$f^{1/\log a}(a) = f^{1/\log b}(b).$$

Thus $f(b) = b^k$ with a suitable $k \geqslant 0$ must hold for all b in B.

In view of (i), for each a_0 in A, prime to γ, we may find $q-1$ elements of A, say $a_1, a_2, \ldots, a_{q-1}$, which are mutually prime, prime to γ and exceed γW. If $q > 2$, we may write, for $i = 0, 1, 2, \ldots, q-1$,

$$\frac{f(a_0) f(a_1) \ldots f(a_{q-1})}{f(a_i)} = \left(\frac{a_0 a_1 \ldots a_{q-1}}{a_i} \right)^k,$$

and if $q = 2$, then $f(a_1) = a_1^k$ and $f(a_0) f(a_1) = (a_0 a_1)^k$. As f is positive, this proves $f(a_i) = a_i^k$.

Finally, we see that for all a in A the condition $(a, \gamma) = 1$ implies $f(a) = a^k$, and since (i) implies that for all a, b in A there is a number γ in A exceeding W and prime to a and b, the theorem follows.

COROLLARY 1. *If a set $A \subset N$ satisfies* (i), (ii) *and* (iii) *with an even q, then A has property* II.

COROLLARY 2. *If $(a, b) = 1$, then the set $\{an + b\}$ (where $n = 0, 1, 2, \ldots$) has property* I.

COROLLARY 3. *The set of all square-free numbers has property* II.

COROLLARY 4. *The set $\{3^m(3n+1)\}$ (where $m = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$) has property* III.

COROLLARY 5. *If $(a, b) = 1$, then the set $\{n\colon a \mid n \vee b \mid n\}$ has property* IV.

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