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A CONSTRUCTION OF RESOLVABLE QUADRUPLE SYSTEMS

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1. Introduction. Let X be a finite set and let $\mathscr{B}$ be a family of subsets (called *blocks*) of X. The pair $(X, \mathscr{B})$ is called a *system*.

Definition 1. The system $(X, \mathscr{B})$ is called a $B[k, \lambda, v]$ *design* if

(i) $|X| = v$,

(ii) $|B| = k$ for each B,

(iii) every pair $\{x, y\} \subset X$ is contained in exactly λ blocks of $\mathscr{B}$.

The number b of blocks in $B[k, \lambda, v]$ is

$$b = \lambda v(v-1)/k(k-1).$$

A necessary condition for the existence of $B[k, \lambda, v]$ is

$$\lambda(v-1) \equiv 0 \pmod{k-1} \quad \text{and} \quad \lambda v(v-1) \equiv 0 \pmod{k(k-1)}.$$

This condition is also sufficient for $k = 3, 4, 5$ with exception of the non-existing $B[5, 2, 15]$.

Definition 2. Let $(X, \mathscr{B})$ be a $B[k, \lambda, v]$ design. A family $\mathscr{B}' \subset \mathscr{B}$ of disjoint blocks covering all elements of X with exception of exactly one element will be called a *parallel class of resolvable blocks*.

Clearly, every parallel class of resolvable blocks in $B[k, \lambda, v]$ consists of $(v-1)/k$ blocks.

Definition 3. A $B[k, \lambda, v]$ design $(X, \mathscr{B})$ is called *resolvable* or an $RB[k, \lambda, v]$ if the family $\mathscr{B}$ can be partitioned into parallel classes. The number l of parallel classes of blocks equals $\lambda v/(k-1)$. A necessary condition for the existence of $RB[k, \lambda, v]$ is

$$(1) \qquad v \equiv 1 \pmod{k} \quad \text{and} \quad \lambda v \equiv 0 \pmod{k-1}.$$

The sufficiency of (1) has been proved in the case of $\lambda = 2$, $k = 3$[1].

In this paper we prove

[1] H. Hanani, *On resolvable balanced incomplete block designs*, Journal of Combinatorial Theory (A) 17 (1974), p. 275-289.

THEOREM 1. *If $v = p_1 p_2 \ldots p_r$, where $p_1, p_2, \ldots, p_r$ are prime numbers (not necessarily distinct) such that $p_i \equiv 1 \pmod 4$ for all $1 \leqslant i \leqslant r$, then $RB[4, 3, v]$ exists.*

In this case ($k = 4$, $\lambda = 3$) condition (1) reduces to $v \equiv 1 \pmod 4$ and we also have $l = v$. Before presenting the construction of $RB[4, 3, v]$ we recall the definition of a resolvable pair design.

Definition 4. Let $X = \{1, 2, \ldots, 2t+1\}$ and let P denote the family of all pairs in X. A system $\{P_1, P_2, \ldots, P_{2t+1}\}$ of disjoint subfamilies of P such that each P_i consists of disjoint pairs with

$$\Big|\bigcup_{\{x,y\}\in P_i} \{x, y\}\Big| = 2t$$

is called a *resolvable pair design*.

2. Main construction. Let $X = \{1, 2, \ldots, 4n+1\}$ and suppose that $v = 4n+1$ satisfies the assumption of Theorem 1. We construct a resolvable pair design on X putting $\{x, y\} \in P_i$ for $1 \leqslant i \leqslant 4n+1$ iff $x+y \equiv i \pmod v$. Clearly, we have

$$\bigcup_{\{x,y\}\in P_i} \{x, y\} = X \setminus \{x_i\},$$

where $x_i = i/2$ or $(i+v)/2$ according to the parity of v. With each $\{x, y\} \in P_i$ we associate an index r_{xy} defined by

$$r_{xy} = \begin{cases} |x-y| & \text{if } |x-y| \leqslant 2n, \\ 4n+1-|x-y| & \text{if } |x-y| > 2n. \end{cases}$$

It is easy to check that r_{xy} takes each value from $\{1, 2, \ldots, 2n\}$ exactly once.

Let the first row of a $(2 \times 2n)$-matrix be equal to $(1, 2, \ldots, 2n)$. Fix an integer s satisfying

$$2 \leqslant s \leqslant 2n \quad \text{and} \quad s^2+1 \equiv 0 \pmod v$$

and define the second row according to the following rule:

if a belongs to the first row, then define b by

$$sa \equiv b \pmod v \quad \text{and} \quad 0 < b \leqslant 4n$$

and put

$$c = \begin{cases} b & \text{if } b \leqslant 2n, \\ v-b & \text{if } b > 2n. \end{cases}$$

The matrix obtained in this way will be denoted by C.

LEMMA 1. *By a suitable permutation of columns every matrix C can be reduced to the form*

$$\begin{pmatrix} a_1 & c_1 & a_2 & c_2 & \ldots & a_n & c_n \\ c_1 & a_1 & c_2 & a_2 & \ldots & c_n & a_n \end{pmatrix},$$

where, for each i and $j \neq i$,

$$a_i < c_i \quad \text{and} \quad \{a_i, c_i\} \cap \{a_j, c_j\} = \emptyset.$$

We omit an easy proof.

In the sequel the submatrices

$$\begin{pmatrix} a_i & c_i \\ c_i & a_i \end{pmatrix}$$

will be denoted by $[a_i \quad c_i]$ and called *pieces* of C.

Now we are in a position to construct an $RB[4, 3, v]$ design $(X, \mathscr{B})$. Fix P_i $(1 \leqslant i \leqslant 4n+1)$ and let $\{x, y\} \in P_i$. We have $r_{xy}^{\bullet} = a$ for some a $(1 \leqslant a \leqslant 2n)$ and let c be the entry of C standing below a. Now we find in P_i a pair $\{z, w\}$ such that $r_{zw} = c$. Thus to every piece of C there correspond two pairs $\{x, y\}$ and $\{z, w\}$, so a quadruple $\{x, y, z, w\}$. We define the i-th parallel class $\mathscr{B}_i$ of $(X, \mathscr{B})$ as the family of all such quadruples obtained from all pairs $\{x, y\} \in P_i$. The element x_i of X not occurring in any of those quadruples equals either $i/2$ or $(i+v)/2$.

THEOREM 2. *$\mathscr{B} = \{\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_v\}$ is a resolvable design $RB[4, 3, v]$.*

For the proof it suffices to show that every pair $\{x, y\} \subset X$ occurs in exactly three quadruples of the system $\mathscr{B}$, i.e. that $\mathscr{B}$ is $B[4, 3, v]$, as the resolvability of the systems follows directly from the construction. To this end we will prove a few lemmas.

Define a $(2 \times 2n)$-matrix M as follows: if $\begin{pmatrix} a \\ c \end{pmatrix}$ is the j-th column of C, then the j-th column of M is of the form $\begin{pmatrix} \tilde{a} \\ \tilde{c} \end{pmatrix}$, where

$$\tilde{a} = |c - a|,$$

$$\tilde{c} = \begin{cases} c + a & \text{if } c + a \leqslant 2n, \\ 4n+1-c-a & \text{if } c + a > 2n. \end{cases}$$

LEMMA 2. *$\tilde{a} \neq 0$ and $\tilde{c} \neq 0$ for every $a \in \{1, 2, \ldots, 2n\}$.*

Proof. Suppose $\tilde{a} = 0$ or $\tilde{c} = 0$; then

$$c - a \equiv 0 \pmod{v} \quad \text{or} \quad a + c \equiv 0 \pmod{v}$$

and

$$a(1-s) \equiv 0 \pmod{v} \quad \text{or} \quad a(1+s) \equiv 0 \pmod{v}.$$

As $a < v$, there exists a prime p dividing $(v, s+1)$ or $(v, s-1)$. Thus $s \equiv \pm 1 \pmod{p}$, which leads to $-1 \equiv s^2 \equiv 1 \pmod{p}$, and $p = 2$, a contradiction.

LEMMA 3. *Every element of the set $\{1, 2, \ldots, 2n\}$ occurs in M exactly twice.*

Proof. By Lemma 1 it is obvious that every entry of M occurs in M at least twice. We shall show that each column of M coincides with some column of C. For any piece $[a \quad c]$ of C we first consider the following case:

$$c \equiv sa \pmod{v}, \qquad c \equiv b \pmod{v} \quad \text{and} \quad a+b > 2n.$$

In this case we have $\tilde{a} = c-a \equiv sa-a \pmod{v}$, so

$$s\tilde{a} \equiv s^2a - sa \pmod{v}.$$

Hence

$$s\tilde{a} \equiv -(a+sa) \pmod{v}$$

and in virtue of $\tilde{c} = -(a+b) \pmod{v}$ we get

$$s\tilde{a} \equiv \tilde{c} \pmod{v}.$$

The proof of the remaining cases, namely,

$$c \equiv -b \pmod{v} \quad \text{and} \quad a+b > 2n,$$

$$c \equiv -b \pmod{v} \quad \text{and} \quad a+b \leqslant 2n,$$

$$c \equiv b \pmod{v} \quad \text{and} \quad a+b \leqslant 2n,$$

is analogous.

Let $[a_1 \quad c_1]$ and $[a_2 \quad c_2]$ be two distinct pieces. Suppose $a_1 > a_2$. By the first part of the proof it suffices to show that $\tilde{a}_1 \neq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{c}_2$. Suppose to the contrary that

(*) $\tilde{a}_1 = \tilde{a}_2$ or (**) $\tilde{a}_1 = \tilde{c}_2$.

In case (*) we get $c_1 - a_1 = c_2 - a_2$. Let

$$c_1 \equiv sa_1 \pmod{v} \quad \text{and} \quad c_2 \equiv sa_2 \pmod{v}.$$

Then

$$a_2 - a_1 - s(a_2 - a_1) \equiv 0 \pmod{v}.$$

Let $a_2 - a_1 = a_3$ or $a_1 - a_2 = a_3$, $a_3 \in \{1, 2, \ldots, 2n\}$. Hence $a_3 - sa_3 \equiv 0 \pmod{v}$, but $sa_3 \equiv \pm c_3 \pmod{v}$, so

$$\tilde{a}_3 \equiv 0 \pmod{v} \quad \text{or} \quad \tilde{c}_3 \equiv 0 \pmod{v},$$

which contradicts Lemma 2.

If $c_1 \equiv -sa_1 \pmod{v}$ and $c_2 \equiv sa_2 \pmod{v}$, then

$$a_1 - a_2 \equiv -s(a_2 + a_1) \pmod{v}.$$

Let $a_1 - a_2 = a_3$ or $a_2 - a_1 = a_3$ and $a_1 + a_2 = a_4$ or $v-(a_1+a_2) = a_4$; then

$$a_3 \equiv \pm c_4 \pmod{v},$$

which is impossible.

The proof of (**) is analogous.

LEMMA 4. *Every pair $\{x, y\}$ in X occurs in at least three quadruples of $\mathscr{B}$.*

Proof. By construction, every pair $\{x, y\}$ belongs to exactly one P_i and as such it occurs in exactly one quadruple of $\mathscr{B}_i$. It remains to show that $\{x, y\}$ is also a subset of two other quadruples in $\mathscr{B}$, so that there exist exactly four elements $z_1, w_1, z_2, w_2 \in X$ such that the sets $\{x, z_1, y, w_1\}$ and $\{x, z_2, y, w_2\}$ belong to $\mathscr{B}$. By construction there exist numbers z and w such that

$$\{x, z\} \in P_j \quad \text{and} \quad \{y, w\} \in P_j \quad \text{for some } j \neq i.$$

Therefore, one of the following two conditions holds:

(I) $x+z = y+w = j$ or $x+z = y+w = j+v$,

(II) $x+z = j$ and $y+w = j+v$ ($x+z = j+v$ and $y+w = j$, respectively).

For the sake of simplicity we may assume, without loss of generality, that $r_{xy} \leqslant n$.

(I) Let $x > y$ and $w > z$. Without any loss of generality we assume that $x-y \leqslant 2n$. In this case we have $x-y = w-z$, whence

$$x-y = r_{xy} = r_{wz}.$$

Now we consider three subcases of (I):

(a) $x > z$ and $w > y$,

(b) $x < z$ and $w > y$,

(c) $x > z$ and $w < y$.

(a) If $x-z > 2n$ and $w-y > 2n$, then

$$\begin{aligned} r_{xy}+r_{yw} &= v-(x-z)+v-(w-y) = 2v-(x-y)-(w-z) \\ &= 2v-r_{xy}-r_{wz} = 2(v-r_{xy}), \end{aligned}$$

but, obviously, $r_{xz}+r_{yw} < 4n$, so $2(v-r_{xy}) < 4n$, which contradicts the assumption $r_{xy} \leqslant 2n$.

If $x-z \leqslant 2n$ and $w-y > 2n$, then

$$\begin{aligned} |r_{xz}-r_{yw}| &= \big|x-z-\big(v-(w-y)\big)\big| = |(x-y)+(w-z)-v| \\ &= |v-(x-y)-(w-z)| = |v-2r_{xy}|, \end{aligned}$$

and since, obviously, $|r_{xy}-r_{yw}| \leqslant 2n$, we have $|v-2r_{xy}| \leqslant 2n$, which contradicts the assumption $r_{xy} \leqslant 2n$.

Clearly, the symmetric case $x-z > 2n$, $w-y \leqslant 2n$ eliminates in the same way. Thus we have $x-z \leqslant 2n$ and $w-y \leqslant 2n$. It follows that

$$\tilde{c} = r_{xz}+r_{yw} = x-z+w-y = r_{xy}+r_{wz} = 2r_{xy}.$$

In cases (b) and (c) we argue as in (a) and obtain $\tilde{c} = 2r_{xy}$ or $\tilde{a} = 2r_{xy}$.

(II) Let us assume $x+z = j$ and $y+w = j+v$ (the symmetric subcase of (II) requires only a graphical change). We may also assume $x < y$ and $z < w$. The case of $y-x \leqslant 2n$ and $w-z \leqslant 2n$ is impossible, since $y-x+$ $+w-z = v$ would then imply $r_{yx}+r_{wz} = v$, which contradicts the definition of r_{xy}. Therefore, we must have

$$y-x \leqslant 2n \quad \text{and} \quad w-z > 2n.$$

Hence

$$(y-x)+(w-z) = v, \quad r_{xy}+(w-z) = v, \quad r_{xy} = v-(w-z) = r_{wz},$$

so $r_{xy} = r_{wz}$.

We consider four subcases

(a) $x > z$ and $w > y$,

(b) $x > z$ and $w < y$,

(c) $x < z$ and $w > y$,

(d) $x < z$ and $w < y$,

and analogously as in (I) we obtain $\tilde{c} = 2r_{xy}$ or $\tilde{a} = 2r_{xy}$. Since by Lemma 3 every element of $\{1, 2, \ldots, 2n\}$ occurs in M exactly twice, there exist exactly two pairs, say $\{z_1, w_1\}$ and $\{z_2, w_2\}$, such that

$$\{x, z_1\}, \{y, w_1\} \in P_{j_1} \quad \text{and} \quad \{x, z_2\}, \{y, w_2\} \in P_{j_2}.$$

Thus the sets $\{x, z_1, y, w_1\}$ and $\{x, z_2, y, w_2\}$, both containing $\{x, y\}$, belong to $\mathscr{B}$.

Since at the beginning of the proof it was stated that there is also a quadruple in $\mathscr{B}_i$ containing $\{x, y\}$, the assertion of Lemma 4 will follow if we show that the three quadruples we constructed are really different. But this can easily be deduced from condition (I) or (II), since each of the three quadruples obeys one of them.

Theorem 2 can now be obtained by means of Lemma 4 and by a simple comparison of the number of pairs in $\mathscr{B}$ and the number of all pairs in X.

Theorem 1 follows immediately from Theorem 2.

Remark. Since there exist $B[5, 1, v]$ designs for $v \equiv 1 \pmod{20}$ or $v \equiv 5 \pmod{20}$, it follows from Hanani's remark (op. cit., p. 285) that there exist also $RB[4, 3, v]$ designs for these v's.

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