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EXTREME SYMMETRIC NORMS ON R^2

BY

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By a *norm* on $\boldsymbol{R}^2$ we mean a functional $N: \boldsymbol{R}^2 \to \boldsymbol{R}_+$ such that

1° $N(u) > 0$ for $u \neq 0$,

2° $N(u+v) \leqslant N(u)+N(v)$,

3° $N(\lambda u) = |\lambda|\, N(u)$, $\lambda \in \boldsymbol{R}$.

A norm N is *symmetric* if $N\big((x, y)\big) = N\big((|x|, |y|)\big)$. We denote by $\mathscr{S}$ the set of all symmetric norms on $\boldsymbol{R}^2$ satisfying the condition $N\big((1, 0)\big) = N\big((0, 1)\big) = 1$. The set $\mathscr{S}$ is convex.

The purpose of this note is to give a characterization of the set of extreme points of $\mathscr{S}$ (denoted by $\operatorname{ex}\mathscr{S}$). This solves the problem (P 1223) posed by Professor A. Pietsch at the Winter School on Functional Analysis in January 1978 (1).

To every norm $N \in \mathscr{S}$ there corresponds a closed symmetric convex set $U(N)$ including points (1, 0), (0, 1), defined by

$$U(N) = \{u \in \boldsymbol{R}^2\colon\ N(u) \leqslant 1\}.$$

The set $U(N)$ will be called the unit ball. If we have a closed symmetric convex set $U \subset \boldsymbol{R}^2$ with $(1, 0), (0, 1) \in U$, then the corresponding norm is defined by the Minkowski functional

$$N_U(u) = \inf\{\lambda > 0\colon\ u/\lambda \in U\}.$$

We put $e_\varphi = (\cos\varphi, \sin\varphi)$. For a set $W \subset \boldsymbol{R}^2$ such that $(0, 0) \in \operatorname{Int} W$ we define the function $g_W\colon [0, 2\pi) \to \boldsymbol{R}_+$ by

$$g_W(\varphi) = \inf\{\lambda > 0\colon\ e_{\varphi/\lambda} \in W\}.$$

Further, for a function $g\colon [0, 2\pi) \to \boldsymbol{R}_+$ we define the set

$$V(g) = \{\lambda e_\varphi\colon\ 0 \leqslant \lambda \leqslant 1/g(\varphi),\ \varphi \in [0, 2\pi)\}.$$

(1) See N. Tomczak-Jaegermann, *Problems on Banach spaces*, Colloq. Math. 45 (1981), pp. 45–47.

For a norm N we have $N(e_\varphi) = g_{U(N)}(\varphi)$ and the set

$$\{e_\varphi/g_{U(N)}\colon\ \varphi \in [0, 2\pi)\} = \{u\colon\ N(u) = 1\} = S(N)$$

is the boundary of $U(N)$ (unit sphere). Suppose a closed bounded set $W \subset \boldsymbol{R}^2$ has the following property: if $u \in W$ and $\lambda \in [0, 1]$, then $\lambda u \in W$. Put

$$\lambda_\varphi = \frac{1}{g_W(\varphi)} = \sup\{\lambda\colon\ \lambda e_\varphi \in W\}.$$

The set W is convex if and only if the Euclidean length of a bisectrix in a triangle generated by the vectors $\lambda_{\psi-\alpha} e_{\psi-\alpha}$, $\lambda_{\psi+\alpha} e_{\psi+\alpha}$ (which is equal to $(2\lambda_{\psi-\alpha}\lambda_{\psi+\alpha}\cos\alpha)/(\lambda_{\psi-\alpha}+\lambda_{\psi+\alpha}))$ is less than or equal to λ_ψ for all $\psi, \alpha \in [0, 2\pi)$. Therefore, the set W is convex if and only if

$$(*) \qquad 2g_W(\varphi)\cos\alpha \leqslant g_W(\varphi+\alpha)+g_W(\varphi-\alpha)$$

for all $\alpha, \varphi \in [0, 2\pi)$. Moreover, the equality in $(*)$ holds if and only if the vectors $e_{\varphi-\alpha}/g_W(\varphi-\alpha)$, $e_\varphi/g_W(\varphi)$, $e_{\varphi+\alpha}/g_W(\varphi+\alpha)$ belong to a common straight line.

Note that since the norm $N \in \mathscr{S}$ is symmetric, the function $g_{U(N)}$ is completely defined by its restriction to $[0, \pi/4]$.

We denote by $N^1((x, y)) = |x|+|y|$ and $N^\infty((x, y)) = \max(|x|, |y|)$ the l^1- and l^∞-norms, respectively. If $N \in \mathscr{S}$, then

$$U(N^1) \subset U(N) \subset U(N^\infty)$$

and

$$g_1 \geqslant g_{U(N)} \geqslant g_\infty,$$

where $g_1 = g_{U(N^1)}$ and

$$g_\infty(\varphi) = \begin{cases} \cos\varphi & \text{for } 0 \leqslant \varphi \leqslant \pi/4, \\ \sin\varphi & \text{for } \pi/4 < \varphi \leqslant \pi/2. \end{cases}$$

LEMMA. *Let $N \in \mathscr{S}$. Then*

$$N^{\max} := 2N - N_{V_0} \in \mathscr{S},$$

where $V_0 = \operatorname{conv} V(2g_{U(N)} - g_\infty)$.

Proof. Obviously, $N^{\max}$ is symmetric and $N^{\max}(e_0) = N^{\max}(e_{\pi/2}) = 1$. Since we consider only symmetric norms (from $\mathscr{S}$), it is sufficient to restrict the proof to vectors (x, y) such that $x, y \geqslant 0$. Put

$$A_1 = \{\lambda e_\varphi\colon\ 0 \leqslant \lambda \leqslant 2g_{U(N)}(\varphi) - \cos\varphi,\ \varphi \in [0, \pi/4]\},$$

$$A_2 = \{\lambda e_\varphi\colon\ 0 \leqslant \lambda \leqslant 2g_{U(N)}(\varphi) - \sin\varphi,\ \varphi \in [\pi/4, \pi/2]\}.$$

The set A_1 is convex. Indeed, it is sufficient to verify inequality (*) for the function $g(\varphi) = 2g_{U(N)}(\varphi) - \cos\varphi$ for all φ and α with $(\varphi \pm \alpha) \in [0, \pi/4]$. We obtain this inequality by using inequality (*) for $g = g_{U(N)}$ ($U(N)$ is convex). Analogously, A_2 is convex (cf. Fig. 1).

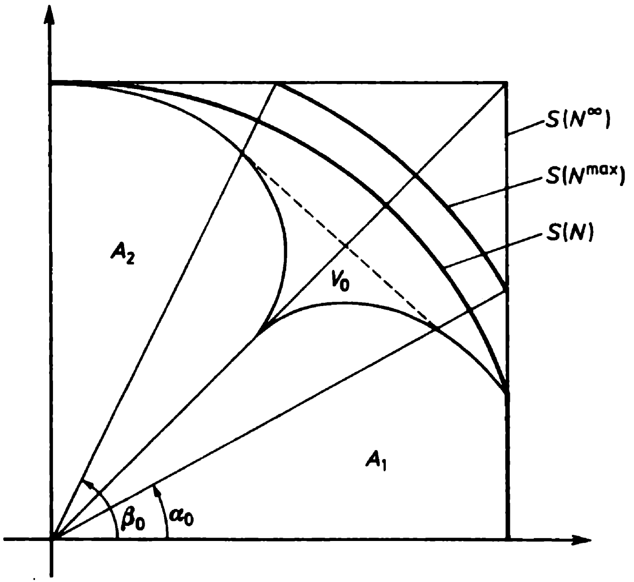

Fig. 1

Since

$$V(2g_{U(N)} - g_\infty) \cap \{(x, y) \colon x, y \geq 0\} = A_1 \cup A_2,$$

it is not hard to see that there exist $\alpha_0 \in [0, \pi/4]$, $\beta_0 \in [\pi/4, \pi/2]$ such that

$$V_0 = \{\lambda e_\varphi \colon \lambda e_\varphi \in A_1,\ \varphi \in [0, \alpha_0]\} \cup \{\lambda e_\varphi \colon \lambda e_\varphi \in A_2,\ \varphi \in [\beta_0, \pi/2]\}$$
$$\cup \operatorname{conv}\{0, \lambda_{\alpha_0} e_{\alpha_0}, \lambda_{\beta_0} e_{\beta_0}\}.$$

Thus

$$g_{V_0}(\varphi) = \begin{cases} c_0 \cos(\varphi - \gamma_0) & \text{for } \varphi \in [\alpha_0, \beta_0], \\ 2g_{U(N)}(\varphi) - g_\infty(\varphi) & \text{for } \varphi \in [0, \alpha_0) \cup (\beta_0, \pi/2], \end{cases}$$

where c_0 and γ_0 satisfy

$$c_0 \cos(\alpha_0 - \gamma_0) = 2g_{U(N)}(\alpha_0) - \cos\alpha_0, \qquad c_0 \cos(\beta_0 - \gamma_0) = 2g_{U(N)}(\beta_0) - \sin\beta_0.$$

Now we have $N^{\max}(e_\varphi) = h(\varphi)$, where $h = 2g_{U(N)} - g_{V_0}$ and $h \geq g_\infty$ ($V(h) \subset U(N)$). The set $V(h)$ is convex. Indeed, since

$$h(\varphi) = g_\infty(\varphi) \quad \text{for } \varphi \in [0, \alpha_0] \cup [\beta_0, \pi/2],$$

we need to check that the set

$$B = \{\lambda e_\varphi \colon 0 \leqslant \lambda \leqslant 1/h(\varphi),\ \varphi \in [\alpha_0, \beta_0]\}$$

is convex. Convexity of B follows from the fact that the function h satisfies $(*)$ for $(\varphi \pm \alpha) \in [\alpha_0, \beta_0]$, which is an easy consequence of the equality $g_{V_0}(\varphi) = c_0 \cos(\varphi - \gamma_0)$.

Since $V(h)$ is convex, $N^{\max}$ is a norm, and the proof is complete.

Let $N = (N_1 + N_2)/2$, $N_i \in \mathcal{S}$, $i = 1, 2$. Obviously, $2g_{U(N)} = g_{U(N_1)} + g_{U(N_2)}$. Since $g_{U(N_i)} \geqslant g_\infty$, we have $g_{U(N_i)} \leqslant 2g_{U(N)} - g_\infty$. It is easy to see that

$$U(N_i) \supset V(2g_{U(N)} - g_\infty) = A_1 \cup A_2.$$

Thus by convexity of $U(N_i)$ we have $U(N_i) \supset V_0$ (i.e., $g_{U(N_i)} \leqslant g_{V_0}$). Therefore,

$$V_0 \subset U(N_i) \subset U(N^{\max}).$$

THEOREM. *Let $N \in \mathcal{S}$. Then $N \in \operatorname{ex} \mathcal{S}$ if and only if*

$$\operatorname{ex} U(N) \subset S(N^\infty).$$

Proof. Let $N = (N_1 + N_2)/2$, $N_i \in \mathcal{S}$, $i = 1, 2$. Then $U(N_{V_0}) \subset U(N_i)$. Obviously,

$$\operatorname{conv}(\operatorname{ex} U(N) \cap S(N^\infty)) \subset U(N_i).$$

If $\operatorname{ex} U(N) \subset S(N^\infty)$, then $U(N) \subset U(N_i)$. Thus $U(N) = U(N_i)$, $N \in \operatorname{ex} \mathcal{S}$.

Let $N \in \operatorname{ex} \mathcal{S}$. Since $2N = N_{V_0} + N^{\max}$, we have

$$N(e_{\alpha_0}) = N_{V_0}(e_{\alpha_0}) = N^{\max}(e_{\alpha_0}) = N^\infty(e_{\alpha_0}) \quad \text{and} \quad N_{V_0}(e_{\beta_0}) = N(e_{\beta_0}).$$

Since $U(N_{V_0}) \subset U(N^\infty)$, the extreme points of $U(N)$ are the following vectors:

$$e_0,\ e_{\alpha_0}/\cos\alpha_0,\ e_{\beta_0}/\sin\beta_0,\ e_{\pi/2}, \ldots$$

Hence $\operatorname{ex} U(N) \subset S(N^\infty)$.

EXAMPLE. Let $N^2((x, y)) = \sqrt{x^2 + y^2}$ be the Euclidean norm. Then

$$2N^2 = N_{V_0} + N^{\max},$$

where

$$N_{V_0}((x, y)) = \begin{cases} \dfrac{\sqrt{7} - 1}{2}(|x| + |y|) & \text{if } \dfrac{4 - \sqrt{7}}{3} \leqslant \left|\dfrac{y}{x}\right| \leqslant \dfrac{4 + \sqrt{7}}{3}, \\ 2\sqrt{x^2 + y^2} - \max(|x|, |y|) & \text{otherwise,} \end{cases}$$

$$N^{\max}((x, y)) = \begin{cases} 2\sqrt{x^2+y^2} - \dfrac{\sqrt{7}-1}{2}(|x|+|y|) & \text{if } \dfrac{4-\sqrt{7}}{3} \leqslant \left|\dfrac{y}{x}\right| \leqslant \dfrac{4+\sqrt{7}}{3}, \\ \max(|x|, |y|) & \text{otherwise.} \end{cases}$$

In this case,

$$\sin\alpha_0 = \frac{\sqrt{7}-1}{2}, \qquad \beta_0 = \pi/2 - \alpha_0.$$

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