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TRIGONOMETRIC INTERPOLATION, IV

BY

R. TABERSKI (POZNAŃ)

1. Preliminaries. Let E be the class of all real functions $f(s)$ Riemann-integrable over any finite interval and such that

$$f(s) = o(s) \quad \text{as } s \to \pm\infty.$$

Given a function $f \in E$ and a positive number l, consider the n-th interpolating polynomial

$$I_n^l(x;f) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l} \right)$$

with nodes

$$s_\nu = s_\nu^{l,n} = \frac{2l\nu}{2n+1} \quad (\nu = 0, \pm 1, \pm 2, \ldots). \tag{1}$$

It can easily be observed that

$$I_n^l(x;f) = \frac{2}{2n+1} \sum_{\nu=-n}^{n} f(s_\nu) D_n^l(s_\nu - x),$$

where

$$D_n^l(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos\frac{k\pi t}{l} = \frac{\sin(2n+1)\dfrac{\pi t}{2l}}{2\sin\dfrac{\pi t}{2l}}.$$

Denote by $\omega_n^l(s)$ the step function which is equal to $2l\nu/(2n+1)$ for $s \in \langle s_{\nu-1}, s_\nu)$ $(\nu = 0, \pm 1, \pm 2, \ldots)$. Taking an arbitrary interval $\langle a, b\rangle$, suppose that

$$s_{\alpha-1} < a \leqslant s_\alpha < s_{\alpha+1} < \ldots < s_\beta < b \leqslant s_{\beta+1}.$$

We shall write

$$\int_a^b g(s)\,d\omega_n^l(s) = \frac{2l}{2n+1}\sum_{\nu=a}^{\beta} g(s_\nu)$$

for any function g defined in $\langle a, b\rangle$ (cf. [6], II, p. 4). In particular,

$$I_n^l(x;f) = \frac{1}{l}\int_{-l}^{l} f(s)D_n^l(s-x)\,d\omega_n^l(s).$$

In Section 3 we shall prove that, for some f's of class E, the last integral tends to $f(x)$ as $l\to\infty$, $n\to\infty$ and $l/n\to 0$. Now a convenient formula for $I_n^l - f$ will be prepared.

Evidently, if $0\leqslant x\leqslant l$, then

$$I_n^l(x;f)-f(x) = \frac{1}{l}\int_{x-l}^{x+l}\{f(s)-f(x)\}D_n^l(s-x)\,d\omega_n^l(s)+$$

$$+\frac{1}{l}\int_{-l}^{x-l}\{f(s)-f(x)\}D_n^l(s-x)\,d\omega_n^l(s)-\frac{1}{l}\int_{l}^{x+l}\{f(s)-f(x)\}D_n^l(s-x)\,d\omega_n^l(s)$$

$$= J_n^l(x)+U_n^l(x)-W_n^l(x).$$

In the case $-l\leqslant x\leqslant 0$, we have

$$I_n^l(x;f)-f(x) = \frac{1}{l}\int_{x-l}^{x+l}\{f(s)-f(x)\}D_n^l(s-x)\,d\omega_n^l(s)+$$

$$+\frac{1}{l}\int_{x+l}^{l}\{f(s)-f(x)\}D_n^l(s-x)\,d\omega_n^l(s)-\frac{1}{l}\int_{x-l}^{-l}\{f(s)-f(x)\}D_n^l(s-x)\,d\omega_n^l(s)$$

$$= J_n^l(x)+\overline{U}_n^l(x)-\overline{W}_n^l(x).$$

Assuming that $0\leqslant a\leqslant x\leqslant b<l$ and $|f(x)|\leqslant K$ (a, b and K mean constants), we have

$$|W_n^l(x)|\leqslant \frac{1}{2l\sin\{\pi(l-x)/2l\}}\int_{l}^{x+l}\{|f(s)|+|f(x)|\}\,d\omega_n^l(s)$$

$$\leqslant \frac{1}{2l\sin\{\pi(l-x)/2l\}}\left\{\int_{l}^{b+l}|f(s)|\,d\omega_n^l(s)+K\int_{l}^{b+l}d\omega_n^l(s)\right\}.$$

The function f is of class E. Therefore, for any $\varepsilon > 0$, there is an $S = S(\varepsilon) > 0$ such that $|f(s)| < \varepsilon s$ if $s > S$. Hence

$$\int_{l}^{b+l} |f(s)|\,d\omega_n^l(s) \leqslant \varepsilon \int_{l}^{b+l} s\,d\omega_n^l(s) < 8bl\varepsilon \quad \text{if } \frac{2l}{2n+1} < b,\ l > S.$$

Consequently,

$$|W_n^l(x)| < \frac{1}{2\sin\{\pi(l-b)/2l\}}\left\{8b\varepsilon + \frac{K}{l}\left(b+\frac{2l}{2n+1}\right)\right\},$$

i.e.,

$$\lim_{l/n\to 0} W_n^l(x) = 0 \text{ uniformly in } x\in\langle a, b\rangle,$$

provided that $l\to\infty$, $n\to\infty$. The summand $U_n^l(x)$ behaves analogously. This implies

$$I_n^l(x;f)-f(x) = J_n^l(x)+o(1) \quad \text{as } l/n\to 0\ (l, n\to\infty), \tag{2}$$

uniformly in $x\in\langle a, b\rangle$.

Considering $-l < a \leqslant x \leqslant b \leqslant 0$ and $|f(x)| \leqslant K$, we obtain

$$\lim_{l/n\to 0} \bar{U}_n^l(x) = 0 = \lim_{l/n\to 0} \bar{W}_n^l(x) \text{ uniformly in } x\in\langle a, b\rangle.$$

Thus relation (2) holds uniformly in x over every finite interval $\langle a, b\rangle$ and, further, investigations may be concentrated upon

$$J_n^l(x) = \frac{1}{l}\int_{x-l}^{x+l} \{f(s)-f(x)\}D_n^l(s-x)\,d\omega_n^l(s). \tag{3}$$

In the sequel, the following analogues of the Bonnet mean-value theorems will be needed:

If the functions $f(s)$ and $g(s)$ are continuous in $\langle a, b\rangle$, and $f(s)$ is non--negative monotonic in $\langle a, b\rangle$, then

$$\left|\int_a^b f(s)g(s)\,d\omega_n^l(s)\right| \leqslant \begin{cases} f(a)\sup\limits_{a<\xi\leqslant b}\left|\int_a^\xi g(s)\,d\omega_n^l(s)\right|, \\ f(b)\sup\limits_{a<\xi\leqslant b}\left|\int_\xi^b g(s)\,d\omega_n^l(s)\right| \end{cases} \tag{4}$$

for non-increasing and non-decreasing f, respectively.

The proof runs as in the classical case.

By $M(u), N(u)$ [respectively, $\bar{M}(u), \bar{N}(u)$] and $M_k(u), N_k(u)$ ($k = 0, 1$) we shall denote the suitable pairs of non-negative convex functions complementary in the sense of Young ([6], I, p. 16 and 170) such that

$$\lim_{u\to 0+}\frac{M(u)}{u}=\lim_{u\to 0+}\frac{N(u)}{u}=0,$$

$$\lim_{u\to\infty}\frac{M(u)}{u}=\lim_{u\to\infty}\frac{N(u)}{u}=\infty,\qquad \text{etc.}$$

For the inverse functions the symbols $M^{-1}(v)$, $N^{-1}(v)$, etc. will be used. The second M-variation $V_M^*(f;a,b)$ will be defined as in Section 1 of [4] and in Section 2 of [5].

2. Analogues of the Riemann-Lebesgue theorem. Throughout this section the real-valued functions $f(s)$ are Riemann-integrable over all finite intervals and subject to further restrictions specified in particular statements.

THEOREM 1. *Suppose that there are three real numbers $h>0$, c^+, c^- and an even function $\varphi(s)$ non-increasing in $\langle h,\infty)$ and such that*

$$\left|\frac{f(s)-c^+}{s}\right|\leqslant\varphi(s),\qquad \left|\frac{f(s)-c^-}{s}\right|\leqslant\varphi(s)$$

for $s\geqslant h$ and $s\leqslant -h$, respectively; moreover,

$$\int_h^\infty\varphi(s)\,ds<\infty.$$

Then, for each real a,b $(a<b)$ and $\delta>0$,

$$\lim_{l/n\to 0}\frac{1}{l}\left(\int_{x-l}^{x-\delta}+\int_{x+\delta}^{x+l}\right)f(s)D_n^l(s-x)\,d\omega_n^l(s)=0$$

uniformly in $x\in\langle a,b\rangle$, provided that $l\to\infty$, $n\to\infty$ (cf. [5], Th. 1).

Proof. Given any $\varepsilon>0$, choose a $\Delta>\max(1,\delta,h-a+1)$ such that

$$\int_{a+\Delta-1}^{\infty}\varphi(s)\,ds<\varepsilon. \tag{5}$$

Taking $x\in\langle a,b\rangle$ and $l>\Delta$, we write

$$J_{l,n}^{+}(x)=\frac{1}{l}\left(\int_{x+\delta}^{x+\Delta}+\int_{x+\Delta}^{x+l}\right)f(s)D_n^l(s-x)\,d\omega_n^l(s)=P+Q.$$

Clearly,

$$Q = \frac{1}{l}\int_{x+\Delta}^{x+l}\{f(s)-c^{+}\}D_n^l(s-x)\,d\omega_n^l(s)+\frac{c^{+}}{l}\int_{x+\Delta}^{x+l}D_n^l(s-x)\,d\omega_n^l(s) = Q_1+Q_2$$

and, by (4),

$$|Q_2| \leqslant \frac{c^{+}}{2l\sin\{\pi\Delta/2l\}}\sup_{x+\Delta<\xi\leqslant x+l}\left|\int_{x+\Delta}^{\xi}\sin(2n+1)\frac{\pi(s-x)}{2l}\,d\omega_n^l(s)\right|$$

$$\leqslant \frac{c^{+}}{2\Delta}\,\frac{2l}{2n+1}\leqslant\frac{c^{+}l}{2n+1}\qquad(n=0,1,2,\ldots).$$

Further,

$$|Q_1| \leqslant \frac{1}{l}\int_{x+\Delta}^{x+l}|f(s)-c^{+}|\,\frac{l}{2(s-x)}\,d\omega_n^l(s)$$

$$\leqslant \frac{1}{2}\left(1+\frac{|x|}{\Delta}\right)\int_{x+\Delta}^{x+l}\varphi(s)\,d\omega_n^l(s)\leqslant\frac{1}{2}(1+|x|)\int_{x+\Delta-1}^{x+l}\varphi(s)\,ds.$$

Putting $\varrho = \max(|a|,|b|)$ and applying (5), we get

$$|Q_1| \leqslant \frac{1}{2}(1+\varrho)\varepsilon\qquad(n=0,1,2,\ldots).$$

To evaluate P, we shall consider two cases.

1° Suppose first that $f(s)$ is continuous in the interval $\langle a-\delta,\, b+\Delta\rangle$. Choose the partition

$$a+\delta = t_0 < t_1 < \ldots < t_j = b+\Delta$$

such that

$$\operatorname*{Osc}_{t_i\leqslant s\leqslant t_{i+1}} f(s) < \frac{\varepsilon\delta}{4\Delta}\qquad(i=0,1,2,\ldots,j-1).$$

In view of (4),

$$|P| \leqslant \frac{1}{2l\sin\{\pi\delta/2l\}}\sup_{x+\delta<\xi\leqslant x+\Delta}\left|\int_{x+\delta}^{\xi}f(s)\sin(2n+1)\frac{\pi(s-x)}{2l}\,d\omega_n^l(s)\right|.$$

Proceeding as in Section 3, 1°, of [5], we obtain $|P|<\varepsilon/2$ if l/n is small enough.

2° In the case where $f(s)$ is Riemann-integrable over $\langle a-\delta, b+\Delta\rangle$, there is a function $g(s)$ continuous in this interval such that

$$\int_{a-\delta}^{b+\Delta} |f(s)-g(s)|\,ds < \frac{\varepsilon\delta}{2}.$$

Assuming that

$$s_{k-1} < a-\delta \leqslant s_k < s_{k+1} < \ldots < s_r < b+\Delta \leqslant s_{r+1},$$

where s_ν denotes the nodes (1), we have

$$\int_{a-\delta}^{b+\Delta} |f(s)-g(s)|\,d\omega_n^l(s) = \frac{2l}{2n+1}\sum_{\nu=k}^{r} |f(s_\nu)-g(s_\nu)|.$$

Hence, by the definition of the Riemann integral,

$$\int_{a-\delta}^{b+\Delta} |f(s)-g(s)|\,d\omega_n^l(s) < \int_{a-\delta}^{b+\Delta} |f(s)-g(s)|\,ds+\frac{\varepsilon\delta}{2}$$

for small l/n.

Writing

$$P = \frac{1}{l}\int_{x+\delta}^{x+\Delta} \{f(s)-g(s)\}D_n^l(s-x)\,d\omega_n^l(s)+\frac{1}{l}\int_{x+\delta}^{x+\Delta} g(s)D_n^l(s-x)\,d\omega_n^l(s)$$

and applying 1°, we obtain

$$|P| \leqslant \int_{a+\delta}^{b+\Delta} |f(s)-g(s)|\frac{1}{2\delta}\,d\omega_n^l(s)+\frac{1}{l}\left|\int_{x+\delta}^{x+\Delta} g(s)D_n^l(s-x)\,d\omega_n^l(s)\right|$$

$$\leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

if l/n is small enough. Thus

$$\lim_{l/n\to 0} J_{l,n}^{+}(x) = 0 \text{ uniformly in } x\in\langle a, b\rangle,$$

and so, by the symmetry, the integral $J_{l,n}^{+}(x)$ can be replaced by

$$J_{l,n}^{-}(x) = \frac{1}{l}\int_{x-l}^{x-\delta} f(s)D_n^l(s-x)\,d\omega_n^l(s).$$

The desired result is now evident.

Remark. If there exist four sequences $\{h_\nu^+\}$, $\{h_\nu^-\}$, $\{c_\nu^+\}$ and $\{c_\nu^-\}$ of real numbers and two sequences $\{\varphi_\nu^+(s)\}$ and $\{\varphi_\nu^-(s)\}$ of non-negative functions non-increasing with $1/|s|$ satisfying the conditions

(i) $0 < h_1^+ \leqslant h_2^+ \leqslant \ldots,\ h_\nu^+ \to \infty, \quad 0 > h_1^- \geqslant h_2^- \geqslant \ldots,\ h_\nu^- \to -\infty,$

(ii)
$$\sum_{\nu=1}^{\infty} |c_\nu^+| + \sum_{\nu=1}^{\infty} |c_\nu^-| < \infty,$$

(iii)
$$\left|\frac{f(s)-c_\nu^+}{s}\right| \leqslant \varphi_\nu^+(s), \qquad \left|\frac{f(s)-c_\nu^-}{s}\right| \leqslant \varphi_\nu^-(s)$$

for $s \in \langle h_\nu^+, h_{\nu+1}^+\rangle$ and $s \in \langle h_{\nu+1}^-, h_\nu^-\rangle$, respectively,

(iv)
$$\sum_{\nu=1}^{\infty} \int_{h_\nu^+}^{h_{\nu+1}^+} \varphi_\nu^+(s)\,ds + \sum_{\nu=1}^{\infty} \int_{h_{\nu+1}^-}^{h_\nu^-} \varphi_\nu^-(s)\,ds < \infty,$$

the conclusion of Theorem 1 remains valid.

THEOREM 2. *Let $f(s)/s$ be of bounded variation over the intervals $(-\infty, -H\rangle$ and $\langle H, \infty)$ for a certain positive H. Then the conclusion of Theorem 1 remains true.*

Proof. As before, consider only the integral $J_{l,n}^+(x) = P+Q$ with $\Delta > \max(1, \delta, H-a)$, $l > \Delta$, $x \in \langle a, b\rangle$.

By the well-known Jordan theorem,

$$\frac{f(s)}{s} = f_1(s) - f_2(s) \qquad (|s| \geqslant H),$$

where $f_k(s)$ $(k = 1, 2)$ are non-negative non-increasing [non-decreasing] in $\langle H, \infty)$ $[(-\infty, -H)]$. Consequently,

$$Q = \sum_{k=1}^{2} (-1)^{k-1} \frac{1}{l} \int_{x+\Delta}^{x+l} s f_k(s) D_n^l(s-x)\, d\omega_n^l(s).$$

Applying (4) twice, we obtain

$$|Q| \leqslant \frac{1}{l} \sum_{k=1}^{2} f_k(x+\Delta) \sup_{x+\Delta<\xi\leqslant x+l} \left| \int_{x+\Delta}^{\xi} s D_n^l(s-x)\, d\omega_n^l(s) \right|$$
$$\leqslant \frac{1}{l} \sum_{k=1}^{2} f_k(x+\Delta) \sup_{x+\Delta<\xi\leqslant x+l} \left| \int_{x+\Delta}^{\xi} (s-x) D_n^l(s-x)\, d\omega_n^l(s) \right| +$$
$$+ \frac{|x|}{l} \sum_{k=1}^{2} f_k(x+\Delta) \sup_{x+\Delta<\xi\leqslant x+l} \left| \int_{x+\Delta}^{\xi} D_n^l(s-x)\, d\omega_n^l(s) \right|$$

$$\leqslant \frac{1}{2}\sum_{k=1}^{2} f_k(x+\Delta) \sup_{x+\Delta<\xi,\eta\leqslant x+l}\left|\int_{\eta}^{\xi} \sin(2n+1)\frac{\pi(s-x)}{2l}\,d\omega_n^l(s)\right| +$$

$$+\frac{|x|}{2\Delta}\sum_{k=1}^{2} f_k(x+\Delta) \sup_{x+\Delta<\xi\leqslant x+l}\left|\int_{x+\Delta}^{\xi} \sin(2n+1)\frac{\pi(s-x)}{2l}\,d\omega_n^l(s)\right|$$

$$\leqslant \frac{l}{2n+1}\{1+\max(|a|,\,|b|)\}\sum_{k=1}^{2} f_k(H) \qquad (l>\Delta,\, n=0,1,2,\ldots).$$

The term P can be estimated as previously. Thus the proof is completed.

Taking into account the pair of functions

$$\Phi_{l,n}^x(t) = \frac{1}{l}\int_x^t D_n^l(s-x)\,d\omega_n^l(s), \qquad \Psi_{l,n}^x(t) = \frac{1}{l}\int_x^t (s-x)D_n^l(s-x)\,d\omega_n^l(s)$$

$(x\in(-\infty,\infty),\, t\in\langle x, x+l\rangle,\, l>\delta)$, and proceeding as in Section 2 of [5], we easily get the following two auxiliary results:

LEMMA 1. *Let*

$$\text{(i)} \qquad M_0(u) = u^q \quad (q>1)$$

or

$$\text{(ii)} \qquad M_0(u) = u\left(\log\frac{2}{u}\right)^{-\gamma} \quad (\gamma>1)$$

or $u\in(0,3/2\rangle$. *If* $l>0$, $n=0,1,2,\ldots$ *and* $x\in(-\infty,\infty)$, *then*

$$|V_{M_0}^*(\Phi_{l,n}^x;\, x, x+l)\leqslant \Lambda,$$

where Λ *is a constant depending only on* q *or* γ, *respectively.*

LEMMA 2. *Given any* $q>1$, *consider* $M_0(u)=u^q$ *for* $u>0$. *Suppose that* $0<l\leqslant C(n+1)^{1-1/q}$ $(C=\text{const},\, n=0,1,2,\ldots)$ *and* $x\in(-\infty,\infty)$. *Then*

$$V_{M_0}^*(\Psi_{l,n}^x;\, x, x+l)\leqslant L,$$

where L *is a constant depending only on* q.

These estimates correspond to that of Lemmas 2 and 3 given in [5]. Also an analogue of Lemma 4 can easily be obtained.

THEOREM 3. *Suppose that* $f(s)$ *is of bounded the second* M*-variation over the intervals* $(-\infty, -H\rangle$ *and* $\langle H,\infty)$ $(H\geqslant 0)$ *when*

$$\text{(i)} \qquad M(u) = u^p \quad (p>1)$$

or

$$\text{(ii)} \qquad M(u) = \exp(-1/u^{a}) \qquad (0 < a < 1/2)$$

for sufficiently small $u > 0$. *Then the conclusion of Theorem* 1 *holds.*

Proof in case (ii). Retain the notation P and Q used as above, consider $M_0(u)$ as in Lemma 1 (ii), with $\gamma < (1-\bar{a})/\bar{a}$ $(0 < a < \bar{a} < 1/2)$, and put

$$\bar{M}(u) = \exp(-1/u^{\bar{a}})$$

for small $u > 0$. Then

$$\frac{1}{N_0^{-1}(1)\bar{N}^{-1}(1)} + \sum_{k=1}^{\infty} M_0^{-1}\left(\frac{1}{k}\right)\bar{M}^{-1}\left(\frac{1}{k}\right) = \sigma < \infty$$

(see [5], Theorem 3). Given any $\varepsilon > 0$, let us choose a $\Delta > \delta$ such that

$$V_{\bar{M}}^{*}(f;\, a+\Delta,\, \infty) < \varepsilon.$$

If

$$(6) \qquad s_{k-1} < x+\Delta \leqslant s_k < s_{k+1} < \ldots < s_m < x+l \leqslant s_{m+1} \qquad (a \leqslant x \leqslant b),$$

where s_ν denotes the nodes (1), the Abel transformation leads to

$$Q = \frac{2}{2n+1}\sum_{\nu=k}^{m} f(s_\nu)D_n^l(s_\nu - x) = \frac{2}{2n+1}\sum_{j=0}^{m-k} f(s_{m-j})D_n^l(s_{m-j}-x)$$

$$= \frac{2}{2n+1}\sum_{j=0}^{m-k-1}\sum_{i=0}^{j}\{f(s_{m-j})-f(s_{m-j-1})\}D_n^l(s_{m-i}-x)+$$

$$+\frac{2}{2n+1}f(s_k)\sum_{i=0}^{m-k} D_n^l(s_{m-i}-x) = Q'+Q''.$$

Putting $\mu = \sup|f(s)|$ $(a+\Delta \leqslant s \leqslant b+2\Delta)$, we have

$$|Q''| \leqslant \frac{\mu}{l}\left|\int_{x+\Delta}^{x+l} D_n^l(s-x)\,d\omega_n^l(s)\right| \quad \text{as} \quad \frac{2l}{2n+1} \leqslant \Delta.$$

Hence, by (4),

$$|Q''| \leqslant \frac{\mu}{2l\sin(\pi\Delta/2l)}\,\frac{2l}{2n+1} \leqslant \frac{\mu l}{(2n+1)\Delta} < \varepsilon \quad \text{for small } l/n.$$

Applying inequality (2) of Section 1 in [3] and above-mentioned Lemma 1, we obtain

$$|Q'| \leqslant \sigma V_{\bar{M}}^{*}(f;\, x+\Delta,\, x+l)\, V_{M_0}^{*}(\Phi_{l,n}^{x};\, x+\Delta,\, x+l)$$

$$\leqslant \sigma\varepsilon V_{M_0}^{*}(\Phi_{l,n}^{x};\, x,\, x+l) < \sigma A\varepsilon.$$

Consequently,

$$|Q| < (1+\sigma\Lambda)\varepsilon \quad \text{and} \quad |P| < \varepsilon \quad \text{for small } l/n,$$

uniformly in $x \in \langle a, b\rangle$ (see the proof of Theorem 1). This implies the required assertion.

THEOREM 4. *Let $f(s)/s$ be of bounded the second M-variation over the intervals $(-\infty, -H\rangle$ and $\langle H, \infty)$ $(H \geqslant 0)$ with $M(u)$ as considered in case* (i) *of Theorem* 3. *Then, under restrictions of Lemma* 2 *and assumption $1/p+1/q > 1$, the conclusion of Theorem* 1 *remains valid.*

Proof. In view of Lemma 1 in [5], for an arbitrary $\varepsilon > 0$, there is a $\Delta > \delta$ such that

$$V_M^*(F; a+\Delta, \infty) < \varepsilon, \quad \text{where } F(s) = f(s)/s.$$

Further, if $x \in \langle a, b\rangle$ and $l > \Delta$, then

$$\begin{aligned} Q &= \frac{1}{l}\int\limits_{x+\Delta}^{x+l} f(s) D_n^l(s-x)\, d\omega_n^l(s) \\ &= \frac{1}{l}\int\limits_{x+\Delta}^{x+l} F(s)(s-x) D_n^l(s-x)\, d\omega_n^l(s) + \frac{x}{l}\int\limits_{x+\Delta}^{x+l} F(s) D_n^l(s-x)\, d\omega_n^l(s) \\ &= T_1 + T_2. \end{aligned}$$

In case (6), the Abel transformation gives

$$\begin{aligned} T_1 &= \frac{2}{2n+1}\sum_{j=0}^{m-k-1}\sum_{i=0}^{j}\{F(s_{m-j}) - F(s_{m-j-1})\}(s_{m-i}-x) D_n^l(s_{m-i}-x) + \\ &\quad + \frac{2}{2n+1} F(s_k)\sum_{i=0}^{m-k}(s_{m-i}-x) D_n^l(s_{m-i}-x) = T_1' + T_1''. \end{aligned}$$

Applying (4) and reasoning as in the proof of Theorem 3, we obtain

$$|T_1''| \leqslant |F(s_k)| \cdot \frac{1}{2} \sup_{x+\Delta \leqslant \xi < x+l} \left| \int\limits_{\xi}^{x+l} \sin(2n+1)\frac{\pi(s-x)}{2l}\, d\omega_n^l(s)\right| < \varepsilon$$

for small l/n if $x \in \langle a, b\rangle$.

By inequality (2) of Section 1 in [3] and by Lemma 2,

$$|T_1'| \leqslant \tilde{\sigma} V_M^*(F; x, x+l)\, V_{M_0}^*(\Psi_{l,n}^x; x, x+l) \leqslant \tilde{\sigma} L\varepsilon$$

if $0 < l \leqslant C(n+1)^{1-1/q}$, $x \in \langle a, b\rangle$, with

$$\tilde{\sigma} = \frac{1}{N_0^{-1}(1) N^{-1}(1)} + \sum_{k=1}^{\infty} M_0^{-1}\left(\frac{1}{k}\right) M^{-1}\left(\frac{1}{k}\right) < \infty.$$

The term T_2 tends to zero with l/n, uniformly in x (cf. Theorem 3). Next we proceed as in the previous proof.

3. Criteria of the Dini, Young and de la Vallée-Poussin type. Retaining notation of Section 1, restrict ourselves to functions $f \in E$ satisfying conditions given in any of Theorems 1-3.

THEOREM 5. *Suppose that, for every* $x \in \langle a, b\rangle$ $(-\infty < a \leqslant b < \infty)$, *there is a function* $\mu_x(t)$ *non-decreasing in an interval* $\langle 0, \eta\rangle$ $(\eta = \eta(x))$ *and such that*

$$|f(x \pm t) - f(x)| \leqslant \mu_x(t) \quad \textit{for } t \in \langle 0, \eta\rangle$$

and

$$\int_0^\eta \frac{\mu_x(t)}{t}\, dt < \infty.$$

Then

$$\lim_{l/n \to 0} I_n^l(x; f) = f(x) \tag{7}$$

provided that $l \to \infty$, $n \to \infty$ *and* $a \leqslant x \leqslant b$.

If, moreover, $\mu_x(t) = \mu(t)$ *is independent of* x *and if*

$$\lim_{\tau \to 0+} \int_0^\tau \frac{\mu(t)}{t}\, dt = 0,$$

then convergence (7) *is uniform in* $\langle a, b\rangle$.

Proof of the first part (pointwise convergence). In view of (2), it is enough to show that $J_n^l(x)$, defined by (3), tends to zero as $l/n \to 0$ $(l, n \to \infty)$.

Taking an arbitrary $\varepsilon > 0$ and any $x \in \langle a, b\rangle$, we can find a positive $\delta \leqslant l$ such that

$$\mu_x(\delta) + \int_0^{2\delta} \frac{\mu_x(t)}{t}\, dt < \frac{\varepsilon}{2\pi}.$$

By an argument similar to that of [6], II, p. 17-18,

$$\left| \frac{1}{l} \int_{x-\delta}^{x+\delta} \{f(s) - f(x)\} D_n^l(s - x)\, d\omega_n^l(s) \right| < \frac{\varepsilon}{2}$$

for small l/n (cf. also the proof of Theorem 7).

Applying any of Theorems 1-3, we obtain

$$\left| \frac{1}{l} \left(\int_{x-l}^{x-\delta} + \int_{x+\delta}^{x+l} \right) \{f(s) - f(x)\} D_n^l(s - x)\, d\omega_n^l(s) \right| < \frac{\varepsilon}{2}$$

if l/n is small enough. Thus the proof is completed.

THEOREM 6. *Let $f(s)$ be of bounded the second M_1-variation over an interval $\langle A, B\rangle$, where*

(i) $$M_1(u) = u^{p_1} \quad (p_1 > 1)$$

or

(ii) $$M_1(u) = \exp(-1/u^{a_1}) \quad (0 < a_1 < 1/2)$$

tor sufficiently small $u > 0$. Then relation (7) *holds at every point of confinuity of f in (A, B).*

If f is continuous at every point x of an interval $\langle a, b\rangle \subset (A, B)$, convergence (7) *is uniform in $\langle a, b\rangle$.*

Proof. Given any $\delta > 0$, we have

$$J_n^l(x) = \frac{1}{l}\left(\int_{x-\delta}^{x} + \int_{x}^{x+\delta}\right)\{f(s)-f(x)\}D_n^l(s-x)\,d\omega_n^l(s)+o(1)$$

as $l/n \to 0$, by any of Theorems 1-3.

Now we apply Lemma 1 and proceed as in [2], p. 292, and [5], Theorem 6 (see also the proof of Theorem 3).

THEOREM 7. *Let $f(s)$, being continuous at x, has a primitive function in some neighbourhood of this point. Write*

$$\psi(t) = \int_0^t \{f(x+u)-f(x)\}\,du \quad \textit{for } t\in(-\infty, \infty)$$

and

$$\chi(t) = \begin{cases} \dfrac{1}{t}\psi(t) & \textit{if } t \neq 0, \\ 0 & \textit{if } t = 0. \end{cases}$$

Suppose $|t\chi'(t)| \leqslant \lambda(t)$ for sufficiently small $|t| \geqslant 0$, where $\lambda(t)$ increases with $|t| \leqslant \tau$ and

$$\int_{-\tau}^{\tau} \frac{\lambda(t)}{|t|}\,dt < \infty.$$

Then relation (7) *holds, whenever $l \to \infty$, $n \to \infty$* (cf. [1], p. 247).

Proof. For an arbitrary $\varepsilon > 0$, we choose a $\delta > 0$ such that

$$\lambda(\delta)+\lambda(-\delta)+\int_{-2\delta}^{2\delta}\frac{\lambda(t)}{|t|}\,dt < \frac{\varepsilon}{\pi}.$$

Evidently, this implies

$$\operatorname*{var}_{-2\delta\leqslant t\leqslant 2\delta} \chi(t) = \int_{-2\delta}^{2\delta} |\chi'(t)|\,dt < \frac{\varepsilon}{\pi}.$$

By the assumption, $\psi'(t) = f(x+t)-f(x)$ for $t\in(-\infty, \infty)$. Hence

$$J_n^l(x) = \frac{1}{l}\int_{x-\delta}^{x+\delta} \psi'(s-x)D_n^l(s-x)\,d\omega_n^l(s)+o(1)$$

$$= \frac{1}{l}\int_{x-\delta}^{x+\delta} \chi(s-x)D_n^l(s-x)\,d\omega_n^l(s)+$$

$$+\frac{1}{l}\int_{x-\delta}^{x+\delta} (s-x)\chi'(s-x)D_n^l(s-x)\,d\omega_n^l(s)+o(1).$$

In view of Theorem 6, which generalizes the corresponding test of Dirichlet-Jordan type, the penultimate integral tends to zero as $l/n \to 0$ $(l, n \to \infty)$.

Write

$$R = \frac{1}{l}\left(\int_{x-\delta}^{x} + \int_{x}^{x+\delta}\right)(s-x)\chi'(s-x)D_n^l(s-x)\,d\omega_n^l(s) = R_1+R_2,$$

and consider the fundamental points (1) such that

$$s_{k-1}\leqslant x < s_k < s_{k+1} < \ldots < s_{k+r} \leqslant x+\delta < s_{k+r+1}.$$

Then, putting $h = 2l/(2n+1)$, we have

$$|R_2| \leqslant \int_{s_k}^{x+\delta} \frac{\lambda(s-x)}{2(s-x)}\left|\sin(2n+1)\frac{\pi(s-x)}{2l}\right| d\omega_n^l(s)$$

$$\leqslant \frac{2l}{2n+1}\,\frac{\lambda(s_k-x)}{2(s_k-x)}(2n+1)\frac{\pi(s_k-x)}{2l} + \frac{2l}{2n+1}\sum_{\nu=k+1}^{k+r}\frac{\lambda(s_\nu-x)}{2(s_\nu-x)}$$

$$\leqslant \frac{\pi}{2}\lambda(h)+h\sum_{i=1}^{r}\frac{\lambda\big((i+1)h\big)}{2ih} \leqslant \frac{\pi}{2}\left\{\lambda(\delta)+\int_0^{2\delta}\frac{\lambda(t)}{t}\,dt\right\}$$

for l/n small enough (cf. [6], II, p. 18). Analogously,

$$|R_1| \leqslant \frac{\pi}{2}\left\{\lambda(-\delta)+\int_{-2\delta}^{0}\frac{\lambda(t)}{|t|}\,dt\right\}.$$

Therefore, if l/n is sufficiently small,

$$|R| \leqslant \frac{\pi}{2}\left\{\lambda(\delta)+\lambda(-\delta)+\int_{-2\delta}^{2\delta}\frac{\lambda(t)}{|t|}\,dt\right\} < \frac{\varepsilon}{2};$$

whence $|J_n^l(x)| < \varepsilon$, and the result follows.

Remark. For f's of class E, Theorem 4 leads to Theorems 5-7 in which the assumptions of Lemma 2 together with $1/p+1/q>1$ are added.

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INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY, POZNAŃ

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