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# QUADRATIC FORMS OVER SPRINGER FIELDS

BY

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A well-known theorem of Springer [5] asserts that the Witt ring $W(k)$ of a field $k$ complete with respect to a discrete valuation is isomorphic to the group ring $W(\bar{k})[h]$ of a 2-element group $h$ over the Witt ring $W(\bar{k})$ of the residue class field $\bar{k}$ (characteristic of $\bar{k}$ is assumed to be different from 2). A plain converse of this theorem is obviously false; for example, for the prime field $F_5$ we have $W(F_5) = W_0[\{1, 2\}]$, where $\{1, 2\}$ is the multiplicative subgroup of $F_5$, and $W_0$ is the subring $\{0, 1\}$ of $W(F_5)$, but $F_5$ is not complete with respect to a discrete valuation.

However, we shall show that if, for a field $k$, the Witt ring $W(k)$ is the group ring of a 2-element group, then there exists a field $K$ which is complete with respect to a discrete valuation and such that $k$ and $K$ are equivalent with respect to quadratic forms in the meaning of [1], that is, their Witt rings are isomorphic.

For example, for $k = F_5$, we can take $K = C((t))$, the formal power series field over the complex numbers.

We also discuss the more general case where $W(k) = W_0[h]$ is the group ring of a finite elementary 2-group $h$ over a subring $W_0$ of $W(k)$. In that case we establish a similar result and, moreover, we find the connection between the number of non-isomorphic quaternion algebras over the field $k$ and the numbers of binary forms and 2-fold Pfister forms belonging to the subring $W_0$.

Notation. For a field $k$ we denote by $k^*$ the multiplicative group of the field, and by $g(k) = k^*/k^{*2}$ the group of square classes. The cardinality $|g(k)|$ of the group will be denoted by $q = q(k)$. The symbol $\langle a_1, \ldots, a_n\rangle$ denotes the class of isometric quadratic forms with the diagonalization $(a_1, \ldots, a_n)$. If a quadratic form $\varphi$ represents $a \in k$ over the field $k$, then we write $\varphi \approx_k a$, or $\varphi \approx a$, and we put

$$D_k\langle a_1, \ldots, a_n\rangle = \{ak^{*2} \in g(k)\colon \langle a_1, \ldots, a_n\rangle \approx_k a\}.$$

The class of isomorphic quaternion algebras containing the algebra

$$\left(\frac{a, b}{k}\right)$$

will be denoted by $(a, b)$.

Suppose that $\pi$ is an element of the field $k$ and $\pi \notin \pm k^{*2}$. We denote by $h_\pi$ any subgroup of $g(k)$ such that

$$g(k) = \{k^{*2}, \pi k^{*2}\} \oplus h_\pi,$$

and by $W_\pi$ the subring of the Witt ring $W(k)$ generated by the forms $\langle a\rangle$, where $ak^{*2} \in h_\pi$.

Definition. For a field $k$ of characteristic different from 2 we say that $k$ is a *Springer field* if there exists an element $\pi \in k^*$, $\pi \notin \pm k^{*2}$, such that the Witt ring $W(k)$ satisfies

$$W(k) = W_\pi[\{k^{*2}, \pi k^{*2}\}],$$

that is, $W(k)$ is the group ring over $W_\pi$ of the 2-element group $\{k^{*2}, \pi k^{*2}\}$. We shall also say that $k$ is *$\pi$-Springer*.

By the main result of [5], any local field $k$ with the prime element $\pi$ and the residue class field of characteristic different from 2 is $\pi$-Springer.

LEMMA 1. *Let $\pi \in k^*$ and $\pi \notin \pm k^{*2}$, where $k$ is a field of characteristic different from 2. The following statements are equivalent:*

(i) *$k$ is $\pi$-Springer.*

(ii) *If $a_1 k^{*2}, a_2 k^{*2} \in h_\pi$, then*

$$D\langle a_1, a_2\pi\rangle = \{a_1 k^{*2}, a_2 \pi k^{*2}\}.$$

(iii) *If $a_1 k^{*2}, a_2 k^{*2} \in h_\pi$, then*

$$D\langle a_1, a_2\rangle \subset h_\pi \quad \text{or} \quad \langle a_1, a_2\rangle = \langle 1, -1\rangle.$$

(iv) *If $a_1 k^{*2}, \ldots, a_n k^{*2} \in h_\pi$ and $\langle a_1, \ldots, a_n\rangle$ is anisotropic, then*

$$D\langle a_1, \ldots, a_n\rangle \subset h_\pi.$$

Proof. (i) $\Rightarrow$ (ii). Let $b$ be any field element represented by $\langle a_1, a_2\pi\rangle$. Then either $bk^{*2} \in h_\pi$ or $b\pi k^{*2} \in h_\pi$. We have $\langle a_1, a_2\pi\rangle = \langle b, a_1 a_2 \pi\rangle$ and, in the first case, from the uniqueness of representation of group ring elements we obtain $\langle a_1\rangle = \langle b\rangle$, i.e., $bk^{*2} = a_1 k^{*2}$. In the second case, $a_1 a_2 b\pi k^{*2} \in h_\pi$, so we must have $bk^{*2} = a_2 \pi k^{*2}$. Thus any field element represented by $\langle a_1, a_2\pi\rangle$ equals $a_1$ or $a_2\pi$ modulo squares.

(ii) $\Rightarrow$ (iii). If $\langle a_1, a_2\rangle \approx b\pi$, where $bk^{*2} \in h_\pi$, then $\langle a_1, -b\pi\rangle \approx -a_2$ and, by (ii), $a_1$ and $-a_2$ belong to the same square class, that is, $\langle a_1, a_2\rangle$ is the hyperbolic plane, as required.

(iii) $\Rightarrow$ (iv). This follows by induction on $n$ from the formula

$$D\langle a_1, \ldots, a_n\rangle = \bigcup\{D\langle a_1, b\rangle : bk^{*2} \in D\langle a_2, \ldots, a_n\rangle\}.$$

(iv) $\Rightarrow$ (i). Clearly, any element $\varphi$ of the Witt ring $W(k)$ is of the form $\varphi = \varphi_1 + \pi\varphi_2$, where $\varphi_1, \varphi_2 \in W_\pi$. It remains to show that this representation is unique or, equivalently, that $W_\pi \cap \pi W_\pi = 0$. Suppose that $\varphi$ is an anisotropic form in $W_\pi \cap \pi W_\pi$. Then

$$\varphi = \langle a_1, \ldots, a_n\rangle = \langle b_1\pi, \ldots, b_n\pi\rangle$$

for suitable $a_i k^{*2}, b_j k^{*2} \in h_\pi$. Thus $\langle a_1, \ldots, a_n\rangle \approx b_1\pi$, contrary to (iv). So we have proved that $W(k) = W_\pi \oplus \pi W_\pi$ and this means that $k$ is $\pi$-Springer.

PROPOSITION. *If a field $k$ is $\pi$-Springer, then its quadratic extension $F = k(\sqrt{\pi})$ is $\sqrt{\pi}$-Springer. Moreover, $k$ and $F$ are equivalent with respect to quadratic forms.*

Proof. We have $D_k\langle 1, -\pi\rangle = \{k^{*2}, -\pi k^{*2}\}$ by Lemma 1, and also $N_{F/k}(\sqrt{\pi}) = -\pi$, whence $g(F) = \{F^{*2}, \sqrt{\pi}F^{*2}\}\oplus h$, where $h$ is the subgroup of $g(F)$ with the same coset representatives as the elements of $h_\pi$ have in $g(k)$ (cf. [2]). First we show that $F$ satisfies (iii) of Lemma 1. Suppose that $a \in F$ and $aF^{*2} \in h$, $a \notin -F^{*2}$, and assume that $\langle 1, a\rangle \approx_F d\sqrt{\pi}$, where $d \in F$, $dF^{*2} \in h$. Then there exist elements $a_1$ and $d_1$ in $k^*$ such that $a_1 k^{*2} \in h_\pi$, $d_1 k^{*2} \in h_\pi$, $a_1 F^{*2} = aF^{*2}$, $d_1 F^{*2} = dF^{*2}$, and $a_1 \notin -k^{*2}$. We also have $\langle 1, a_1\rangle \approx_F d_1\sqrt{\pi}$, so there exist $x, y, z, w \in k$ such that

$$(x+y\sqrt{\pi})^2 + a_1(z+w\sqrt{\pi})^2 = d_1\sqrt{\pi}.$$

Then

$$x^2+\pi y^2+a_1 z^2+a_1\pi w^2 = 0.$$

Putting $b_1 = x^2+a_1 z^2$ and $b_2 = y^2+a_1 w^2$ we have $b_1 \neq 0$ and $b_2 \neq 0$, and the form $\langle b_1, b_2\pi\rangle$ turns out to be isotropic over $k$. But $k$ is $\pi$-Springer, whence, by Lemma 1 (iii), $b_1 k^{*2}, b_2 k^{*2} \in h_\pi$, and so, by Lemma 1 (ii), $\langle b_1, b_2\pi\rangle$ is anisotropic, a contradiction. Thus we have proved that $D_F\langle 1, a\rangle \subset h$. Now, if $a, b \in F$ and $aF^{*2}, bF^{*2} \in h$, then

$$D_F\langle a, b\rangle = aF^{*2}\cdot D_F\langle 1, ab\rangle \subset h \quad \text{for } ab \notin -F^{*2},$$

that is, statement (iii) of Lemma 1 is satisfied. Hence $F$ is $\sqrt{\pi}$-Springer.

Now we show that $k$ and $F$ are equivalent with respect to quadratic forms. Clearly, there is a group isomorphism $\varphi\colon g(k) \to g(F)$ such that $\varphi(\pi k^{*2}) = \sqrt{\pi}F^{*2}$ and $\varphi(ak^{*2}) = aF^{*2}$ for any coset $ak^{*2}$ in $h_\pi$. It is easy to observe that any such isomorphism sends $-k^{*2}$ into $-F^{*2}$. To prove that $k$ and $F$ are equivalent with respect to quadratic forms it remains to show that

$$\varphi(D_k\langle 1, a\rangle) = D_F\langle 1, a\rangle \quad \text{and} \quad \varphi(D_k\langle 1, a\pi\rangle) = D_F\langle 1, a\sqrt{\pi}\rangle$$

for $a \in k^*$, $ak^{*2} \in h_\pi$ (cf. [1], Proposition 2.2). The second equality is immediate:

$$\varphi(D_k\langle 1, a\pi\rangle) = \varphi(\{k^{*2}, a\pi k^{*2}\}) = \{F^{*2}, a\sqrt{\pi}F^{*2}\} = D_F\langle 1, a\sqrt{\pi}\rangle.$$

Also $\varphi(D_k\langle 1, a\rangle) \subset D_F\langle 1, a\rangle$ follows at once from the definition of $\varphi$ and from Lemma 1 (iii). The converse inclusion is trivial if $a \in -k^{*2}$, so assume that $a \notin -k^{*2}$ and $\langle 1, a\rangle \approx_F d$. Then from the first part of the proof it follows that $dF^{*2} \in h$, and hence there exists $d_1 \in k^*$ such that $d_1F^{*2} = dF^{*2}$, $d_1k^{*2} \in h_\pi$, and $d_1 \notin -k^{*2}$. If

$$(x+y\sqrt{\pi})^2+a(z+w\sqrt{\pi})^2 = d_1, \quad \text{where } x, y, z, w \in k,$$

then we have

$$\langle b_1, b_2\pi\rangle \approx_k d_1 \quad \text{for } b_1 = x^2+az^2, b_2 = y^2+aw^2.$$

Hence $d_1k^{*2} = b_1k^{*2}$ by Lemma 1 (ii). But $\langle 1, a\rangle \approx_k b_1$, whence also $\langle 1, a\rangle \approx_k d_1$, as required.

Now we can prove the main result of the paper.

THEOREM 1. *If $k$ is a Springer field, then there exists a field $K$ which is complete with respect to a discrete valuation and such that $k$ and $K$ are equivalent with respect to quadratic forms.*

Proof. Suppose that $\pi \in k^*$, $\pi \notin \pm k^{*2}$, and $k$ is $\pi$-Springer. In the algebraic closure of $k$ we define the sequence

$$\pi_0 = \pi, \pi_1 = \sqrt{\pi_0}, \ldots, \pi_{i+1} = \sqrt{\pi_i}, \ldots$$

and the corresponding tower of fields

$$k_0 = k, k_1 = k_0(\pi_1), \ldots, k_{i+1} = k_i(\pi_{i+1}), \ldots$$

It follows from the Proposition that each field $k_i$ is $\pi_i$-Springer and any two of them are equivalent with respect to quadratic forms. Put $L = \bigcup k_i$ and $K = L((t))$, the formal power series field over $L$. From [2] it follows that $g(L)$ consists of square classes which have the same coset representatives as the square classes in the subgroup $h_\pi$ of $g(k)$. By [7], Lemma 4.2, we obtain $g(K) = h \oplus \{K^{*2}, tK^{*2}\}$, where $h$ and $g(L)$ and, therefore, $h$ and $h_\pi$, have the same coset representatives. We want to prove that $k$ and $K$ are equivalent with respect to quadratic forms. Let us consider the group isomorphism $\psi$: $g(k) \to g(K)$ such that $\psi(\pi k^{*2}) = tK^{*2}$ and $\psi(ak^{*2}) = aK^{*2}$ for any $ak^{*2} \in h_\pi$. Since $\pi \notin \pm k^{*2}$, we must have $-k^{*2} \in h_\pi$, and so $\psi(-k^{*2}) = -K^{*2}$. According to [1], Proposition 2.2, it suffices to show that, for any $a \in k^*$,

$$\psi(D_k\langle 1, a\rangle) = D_K\langle 1, a\rangle.$$

This is obvious for $a \in -k^{*2}$ and to get the result for the remaining values of $a$ we distinguish two cases. First, suppose that $ak^{*2} \in h_\pi$, $a \notin -k^{*2}$. Then $D_k\langle 1, a\rangle \subset h_\pi$ by Lemma 1 (iii), and so

$$\psi(D_k\langle 1, a\rangle) \subset D_K\langle 1, a\rangle \subset h.$$

The other inclusion is proved as follows. If $\langle 1, a\rangle \approx_K d$, then $dK^{*2} = d_1K^{*2}$, where $d_1 \in k^* \subset L^*$. Hence, by [7], Lemma 4.2, we have

$\langle 1, a\rangle \approx_L d_1$, i.e., $x^2+ay^2 = d_1$ with $x, y \in L$. Then for a suitable index $i$ we have $x, y \in k_i$, that is, $\langle 1, a\rangle \approx_{k_i} d_1$. By the Proposition we have then $\langle 1, a\rangle \approx_{k_{i-1}} d_1$ and, by induction, $\langle 1, a\rangle \approx_k d_1$. This proves the equality $\psi(D_k\langle 1, a\rangle) = D_K\langle 1, a\rangle$ in the first case. The second possible case is $ak^{*2} = b\pi k^{*2}$, where $bk^{*2} \in h_\pi$. Here we have

$$\psi(D_k\langle 1, b\pi\rangle) = \psi(\{k^{*2}, b\pi k^{*2}\}) = \{K^{*2}, btK^{*2}\} = D_K\langle 1, bt\rangle,$$

as required. Thus $k$ and $K$ are equivalent with respect to quadratic forms, and since formal power series fields are well known to be complete with respect to the standard valuation, the theorem is proved.

COROLLARY 1. *If $k$ is $\pi$-Springer field and $K = L((t))$, where $L$ is the field constructed in the proof above, then the Witt ring $W(L)$ is isomorphic to the subring $W_\pi$ of $W(k)$. In particular, if $k$ is a local field, then $L$ is equivalent with respect to quadratic forms to the residue class field $\bar{k}$, provided the latter has characteristic different from 2.*

Example of a Springer field. If $k$ is a field such that $D_k\langle 1, 1\rangle = \{\pm k^{*2}\}$ and $-1 \notin k^{*2}$, then $K = k(\sqrt{-1})$ is a Springer field. More precisely, if $\pi \in K$ and $N_{K/k}(\pi) = -1$, then $K$ is $\pi$-Springer. This can be proved by showing that $K$ satisfies (ii) of Lemma 1. (This result complements Proposition 5.14 of [1].)

Now we will sketch a generalization of Theorem 1.

THEOREM 2. *Let $k$ be a field of characteristic different from 2, let $g(k) = h \oplus h_0$, where $|h| = 2^n$, $n \geq 1$, and assume that $-k^2 \notin h$ whenever $-k^{*2} \neq k^{*2}$. Further, let $W_0$ be the subring of the Witt ring $W(k)$ generated by the forms $\langle a\rangle$, where $ak^{*2} \in h_0$. If $W(k)$ is the group ring $W_0[h]$, then there exists a field $k_0$ such that $k$ and $k_0((t_1))\ldots((t_n))$ are equivalent with respect to quadratic forms.*

Proof (by induction on $n$). The case $n = 1$ has been settled in Theorem 1. Suppose that $n \geq 2$ and put

$$h = \{k^{*2}, \pi k^{*2}\} \oplus h_1 \quad \text{and} \quad W_1 = W_0[h_1].$$

Then $W_1$ is the subring of $W(k)$ generated by the forms $\langle a\rangle$, where $ak^{*2} \in h_1 \oplus h_0$ and $W(k)$ is the group ring $W_1 + \pi W_1$. Hence $k$ is $\pi$-Springer, and so there exists a field $L$ such that $k$ and $L((t))$ are equivalent with respect to quadratic forms. Also $g(L) = h_1' + h_0'$, where the cosets in $h_1'$, $h_0'$ and $h_1$, $h_0$ have the same representatives, respectively, and $|h_1| = 2^{n-1}$. Corollary 1 implies that $W(L)$ and $W_1$ are isomorphic, and hence $W(L)$ is the group ring $W_0'[h_1']$, where the subring $W_0'$ of $W(L)$ is generated by the forms $\langle a\rangle$ with $aL^{*2} \in h_0'$. By induction, there is a field $k_0$ such that $L$ and $k_0((t_1))\ldots((t_{n-1}))$ are equivalent, and hence so are $k$ and $k_0((t_1))\ldots((t_n))$, which completes the proof.

We end this paper with some results concerning the number $Q(k)$ of non-isomorphic quaternion algebras over a field $k$ which is $\pi$-Springer. Apart from the standard theory (cf. [3]) we use here Theorems 1 and 2 and the results of [6]. From [3], 57:8, we conclude that $Q(k) = p_2(k)$, the number of isometry classes of 2-fold Pfister forms over $k$. Since any isomorphism of Witt rings $W(k_1) \to W(k_2)$ sends bijectively the set of 2-fold Pfister forms over $k_1$ onto the set of 2-fold Pfister forms over $k_2$, we obtain the following

LEMMA 2. *If $k_1$ and $k_2$ are two fields equivalent with respect to quadratic forms, then $Q(k_1) = Q(k_2)$.*

THEOREM 3. *Suppose that $k$ is a $\pi$-Springer field and let $Q_\pi(k)$ and $N_\pi(k)$ be the numbers of 2-fold Pfister forms and binary quadratic forms belonging to the subring $W_\pi$ of $W(k)$, respectively. If the number $q$ of square classes of the field $k$ is finite, then so are $Q(k)$, $Q_\pi(k)$, $N_\pi(k)$ and, moreover,*

$$Q(k) = Q_\pi(k) + N_\pi(k) - 1.$$

Proof. From the proof of Theorem 1 we know that there exists a field $L$ such that $k$ and $L((t))$ are equivalent with respect to quadratic forms, and rings $W(L)$ and $W_\pi$ are isomorphic. Hence $N_\pi(k)$ equals the number $N_2(L)$ of binary forms in $W(L)$, and $Q_\pi(k) = p_2(L) = Q(L)$. Now Lemma 2 and [6], 1.6, give the result.

COROLLARY 2. *Let $k$ be a field complete with respect to a discrete valuation and $\bar{k}$ its residue class field. Denote by $N_2(\bar{k})$ the number of non-isometric binary forms over $\bar{k}$. Then, if $q(k)$ is finite, then so are $Q(k)$, $Q(\bar{k})$, $N_2(\bar{k})$, and*

$$Q(k) = Q(\bar{k}) + N_2(\bar{k}) - 1.$$

Using Theorem 2 and [6], 1.8, we obtain the following result:

COROLLARY 3. *Let $k$ be a field of characteristic different from 2 and let $g(k) = h \oplus h_0$, where $|h| = 2^n$, $n \geqslant 1$, and $-k^{*2} \notin h$ whenever $-k^{*2} \neq k^{*2}$. Let $W_0$ be the subring of $W(k)$ generated by the forms $\langle a \rangle$, $ak^{*2} \in h_0$, and denote by $Q_0(k)$ and $N_0(k)$ the numbers of 2-fold Pfister forms and binary quadratic forms in the subring $W_0$, respectively. If $W(k) = W_0[h]$ is the group ring over $W_0$ of the group $h$ and $q(k) < \infty$, then $Q(k)$, $Q_0(k)$, $N_0(k)$ are finite and*

$$Q(k) = Q_0(k) + (2^n - 1)N_0(k) + [\tfrac{1}{3}(2^{2n-1}+1) - 2^{n-1}]q^2 - 2^n + 1.$$

Remark. The referee has kindly pointed out that, in the case where $k$ is the field $R$ of real numbers, the number of non-isomorphic quaternion algebras over the fields $R((t_1)) \ldots ((t_n))$ coincides with the number of 4-dimensional composition algebras over the field $R((t_1, \ldots, t_n))$, which in both cases equals $\tfrac{1}{3}(2^{2n+1}+1)$ (cf. [4]).

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