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OMITTING CARDINALS IN TAME SPACES

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1. Introduction. A topological space X is said to *omit* a cardinal $\varkappa$ if its cardinality is greater than $\varkappa$ but it has no closed subspaces of cardinality $\varkappa$. Perhaps the best-known example that does not require special axioms is the Stone–Čech compactification of the integers: it omits every infinite cardinal $< 2^{\mathfrak{c}}$ ([5], 9.12). On the other hand, the first-named author [7] has shown that if the Generalized Continuum Hypothesis (GCH) is assumed, no compact T_2-space can omit both $\varkappa^+$ and $\varkappa^{++}$ for any cardinal $\varkappa$. Other results on omitting cardinals can be found in [6] and [9].

In this paper we will study which cardinals may or may not be omitted by regular (this includes Hausdorff) "tame" spaces. *Tame* is our temporary term for any space X such that $|\bar{Y}| \leqslant 2^{|Y|}$ for any subset Y of X. (We use the notation $|A|$ for the cardinality of A.) Tame spaces are easily seen to include all first countable regular spaces, all linearly ordered spaces, and all regular scattered spaces (Section 2). Even for them, the question of whether a cardinal can be omitted depends very much upon the description of the cardinal and the axioms one assumes. One illustration is provided by the real line: every infinite closed subspace is either of cardinality ω or $\mathfrak{c}(= 2^{\omega})$, so that if one does not assume CH, even it will omit one or more cardinals. Nor does it help to confine our attention to compact scattered spaces, as Example 1 in Section 2 shows. On the other hand, we will have some general results about when certain cardinals cannot be omitted.

Our interest in these matters was stimulated by a manuscript [11] of Monk, in which Problem 44 reads:

If A is a hereditarily atomic Boolean algebra, does it have homomorphic images of every cardinality $\leqslant |A|$?

Since the weight of the Stone space equals the cardinality of A for any Boolean algebra A, and since weight equals cardinality for compact Hausdorff scattered spaces, and since a Boolean algebra is hereditarily atomic (the word "superatomic" is synonymous) if and only if its Stone space is scattered, Monk's question is equivalent to:

If X is a compact Hausdorff scattered space, must X have closed subspaces of every smaller cardinality?

While we do not yet have a complete answer to this question, we do have a negative answer in every model of $\neg$CH (Example 1) and also in every model in which the covering lemma holds with respect to the core model K (Example 2). Thus, if there is an affirmative answer to Monk's problem in some model of ZFC, it will have an inner model with many measurable cardinals.

However, it is also worth noting that some cardinals are easier to omit than others. The most fundamental unsolved problem seems to be:

PROBLEM 1 (**P 1363**). Is there any model of set theory in which $\mathfrak{c}$ is omitted by some compact T_2 scattered space?

Our set-theoretic notation will be standard. In particular, we identify a cardinal $\varkappa$ with the set of all ordinals $< \varkappa$, and exponentiation is always cardinal exponentiation. We use the notation

$$\varkappa^{<\lambda} = \sup\{\varkappa^\gamma : \gamma < \lambda\}.$$

Our notation for cardinal functions on spaces will be as in [8]. In particular, if $p \in X$, $\chi(p, X)$ and $\psi(p, X)$ denote the character and pseudo-character of p in X: the former is the least cardinality of a local base at p in X, the latter the least number of open subspaces of X whose intersection is p. As is well known, $\chi(p, X) = \psi(p, X)$ whenever X is compact T_2.

From now on, "space" will always mean "T_2-space".

2. Scattered spaces. A space is called *scattered* (or *dispersed*) if every subspace has a (relatively) isolated point.

NOTATION. Given a space X, let $I(X) = I_0(X)$ denote the set of isolated points of X. If α is an ordinal and I_β has been defined for all $\beta < \alpha$, let $I_\alpha(X)$ be the set of isolated points of

$$X \setminus \bigcup \{I_\beta(X) : \beta < \alpha\}.$$

It is a standard exercise to show that a space X is scattered if and only if each point of X is in $I_\alpha(X)$ for some α. Of course, this ordinal α is uniquely determined for each point $x \in X$. We denote it by $i(x)$.

LEMMA 1. *Every scattered regular space is tame.*

Proof. Since every subspace of a scattered space is scattered, it suffices to show that $|X| \leq 2^{|I(X)|}$ for each regular scattered X. For each point x of X let U_x be an open neighborhood of x such that

$$\bar{U}_x \setminus \{x\} \subset \bigcup \{I_\beta(X) : \beta < i(x)\}.$$

Then no two sets U_x meet $I_0(X) = I(X)$ in the same set, and distinct points give distinct sets U_x.

THEOREM 1. *If X is a scattered regular space and $\varkappa^{<\varkappa} = \varkappa$, then X does not omit $\varkappa$.*

Proof. If $|X| > \varkappa$, then $I(X) \geq \varkappa$. If X omits $\varkappa$, so does every subspace in the closure of $\varkappa$ isolated points. Thus we may assume $|I(X)| = \varkappa$. We consider two cases:

Case 1. *For all* $\alpha < \varkappa$, $|I_\alpha(X)| \leqslant \varkappa$.

In this case, $I_\varkappa(X)$ is nonempty. But if $p \in I_\varkappa(X)$, then there is a closed neighborhood N of p such that $N \setminus \{p\}$ is a subset of $\bigcup \{I_\alpha(X): \alpha < \varkappa\}$, which is of cardinality $\varkappa$. Also, it is an elementary fact that every neighborhood of p meets $I_\alpha(X)$ for all $\alpha < \varkappa$. Thus $|N| = \varkappa$, contradicting the claim that X omits $\varkappa$.

Case 2. *For some ordinal* $\alpha < \varkappa$, $|I_\alpha(X)| > \varkappa$.

Let α be the least such ordinal, so that $|I_\beta(X)| \leqslant \varkappa$ for all $\beta < \alpha$. For each point $p \in I_\alpha(X)$, let F_p be a closed neighborhood of p meeting $I_\alpha(X)$ in $\{p\}$ and contained in $\bigcup \{I_\beta(X): \beta \leqslant \alpha\}$. By the assumption on X, $|F_p| < \varkappa$, and so

$$|F_p \cap I_0(X)| < \varkappa \quad \text{for all } F_p.$$

But if $p \neq q$, $F_p \cap I_0(X) \neq F_q \cap I_0(X)$, and so $I_0(X)$ has more than $\varkappa$ distinct subsets of cardinality $< \varkappa$, contradicting the assumption $\varkappa^{<\varkappa} = \varkappa$.

Corollary 1. *If* GCH *holds, no regular scattered space omits any regular cardinal.*

As a matter of fact, GCH is equivalent to the axiom that $\varkappa^{<\varkappa} = \varkappa$ for all regular cardinals $\varkappa$. Moreover, if ever $\lambda < \varkappa < 2^\lambda$, then $\varkappa^{<\varkappa} \geqslant 2^\lambda$, and this last inequality might even be strict, as in models where $\omega_2 < 2^\omega < 2^{\omega_1}$ ($\lambda = \omega$, $\varkappa = \omega_2$). There are even models where ω is the only infinite cardinal $\varkappa$ that satisfies $\varkappa^{<\varkappa} = \varkappa$, so it is natural to ask what happens in Theorem 1 when this assumption is dropped. For cardinals between ω and 2^ω we have a very satisfactory answer.

Example 1. Without using any set-theoretic assumptions beyond the axiom of choice, Eric van Douwen showed that there is a locally compact, locally countable (hence scattered) topology on $\boldsymbol{R}$ that is finer than the usual topology, and has the property that any subset of $\boldsymbol{R}$ having $\mathfrak{c}$ points in its usual closure, also has $\mathfrak{c}$ points in its closure in the new topology. (Descriptions of such spaces may be found in [12] and [15].) Thus $\boldsymbol{R}$ with this finer topology omits every cardinal strictly between ω and 2^ω, if such exist. The same applies to the one-point compactification of this space, which thus provides a negative answer to Monk's problem in any model of CH.

The case of higher cardinals is not so clear. In [9], one of us describes a model $M(\omega_1, \lambda)$ in which every compact space of character ω_1 omits every cardinal strictly between ω_1 and $2^{\omega_1} = \lambda$; by making λ large enough we can put arbitrarily many omitted cardinals in between. The construction generalizes to give models $M(\varkappa, \lambda)$ for all $\varkappa$ in which $\varkappa$ behaves like ω_1 above. Among the compact spaces of character $\varkappa$ and cardinality $2^\varkappa$ there are some tame spaces like ${}^{\varkappa}2$ with the lexicographical order. This space can be modified like $\boldsymbol{R}$ in Example 1 to give scattered spaces of cardinality $2^\varkappa$ which omit all cardinals between $\varkappa$ and $2^\varkappa$.

There are a number of drawbacks to the examples we have just described. First, they are not absolute, like $\boldsymbol{R}$: we do not know whether every model of set theory has a compact space which omits every cardinal between $\varkappa$ and $2^{\varkappa}$, let alone compact tame or compact scattered ones. Second, the models $M(\varkappa, \lambda)$ all utilize the collapsing of large (inaccessible) cardinals, so we do not know whether the results can be shown consistent by merely assuming the consistency of ZFC. Third, the models are distinct for different $\varkappa$ and they "satisfy GCH up to $\varkappa$", so, in particular, we do not know of any where $\mathfrak{c}$ is omitted by a compact tame space. And fourth, the scattered examples we have been able to obtain by modifying ${}^{\varkappa}2$ are not locally compact, and so they do not provide any new answers to Monk's question.

In an earlier draft to this paper, we posed the problem of whether there is a model of ZFC in which a cardinal of the form $2^{\varkappa}$ is omitted by some regular scattered space; this has since been answered affirmatively. (Note that, because of the third drawback, the foregoing examples do not answer this question.)

3. Orderable and compact examples. A space X with a total order $\leqslant$ is said to be *generalized orderable* if it has a base consisting of intervals. By the method of Dedekind cuts it is easy to show that every generalized orderable space is tame. We also have a result like Theorem 1:

LEMMA 2. *If X is a generalized orderable space and $\varkappa^{<\varkappa} = \varkappa$, then X does not omit $\varkappa$.*

Proof. Suppose X is a generalized orderable space of cardinality $> \varkappa$. If X has a point p such that $\chi(p, X) \geqslant \varkappa$, then X has a well-ordered (or reverse well-ordered) sequence of cardinality $\varkappa$, and hence a closed subspace of that cardinality. So suppose $\chi(p, X) < \varkappa$ for all $p \in X$. Let Y be a subset of X of cardinality $\varkappa$. Each point in the closure of Y is the supremum (or infimum) of a well-ordered sequence from Y, of cardinality $< \varkappa$. Since $\varkappa^{<\varkappa} = \varkappa$, it follows that $|\bar{Y}| = \varkappa$.

The proof of the next result is similar, but more complicated. For it, we use the theorem ([8], 2.5 and the following remark) that, in a space X, the set of points that satisfy $\chi(p, X) \leqslant \lambda$ is of cardinality $\leqslant d(X)^{\lambda}$.

THEOREM 2. *If X is compact and tame, $\varkappa^{<\varkappa} = \varkappa$, and $2^{\varkappa} < 2^{(\varkappa^+)}$, then X does not omit $\varkappa$.*

Proof. If $|X| > \varkappa$, then $d(X) \geqslant \varkappa$. If X omits $\varkappa$, then we may assume $d(X) = \varkappa$ as in the proof of Theorem 1. Let

$$H = \{p \in X: \chi(p, X) < \varkappa\}.$$

By the above remarks, $|H| \leqslant \varkappa^{<\varkappa} = \varkappa$. Again we consider two cases.

Case 1. *There is a point p of X such that $\chi(p, X) = \varkappa$.*

By compactness this is equivalent to $\psi(p, X) = \varkappa$, and so there is a well-ordered net $\langle p_\xi: \xi < \varkappa\rangle$ converging to p: Let $\{U_\xi: \xi < \varkappa\}$ be a base of open neighborhoods of p and let

$$p_\xi \in \bigcap \{U_\alpha: \alpha < \xi\}.$$

By $\psi(p, X) = \varkappa$ we can make the p_ξ all distinct, and every neighborhood of p contains all but $< \varkappa$ of these points. Thus all points in the closure of $A = \{p_\xi = \xi < \varkappa\}$, except perhaps for p, are in the closure of some subset of A of cardinality $< \varkappa$. By tameness and $\varkappa^{<\varkappa} = \varkappa$, we have $|\bar{A}| = \varkappa$.

Case 2. *For each* $p \in X \backslash H$, $\chi(p, X) > \varkappa$.

If p is such a point, there is an open $F_\varkappa$-set containing H and missing p: simply expand each point of H to a cozero set that misses p. Thus X has a nonempty closed $G_\varkappa$-set $F \subset X \backslash H$. For all $q \in F$, $\chi(q, X) \geqslant \varkappa^+$. This implies $\psi(q, F) \geqslant \varkappa^+$ for all $q \in F$. Since F is compact, $|F| \geqslant 2^{\varkappa^+}$ by the Čech–Pospíšil theorem ([8], 3.16). But since X is a tame and $d(X) = \varkappa$, $|X| \leqslant 2^\varkappa$, contradicting $2^\varkappa < 2^{\varkappa^+}$.

PROBLEM 2 (**P 1364**). Can the hypothesis $2^\varkappa < 2^{\varkappa^+}$ be dropped? From the proof of Theorem 2 it follows that if there is a counterexample, there is one such that no point is the intersection of $\leqslant \varkappa$ open sets.

4. Countably compact, first countable spaces. The results in this section shed further light on Monk's question, especially in the case of singular cardinals. The following lemma and its proof are obvious generalizations of Theorem 1 in [10].

LEMMA 3. *Let λ and $\varkappa$ be cardinal numbers such that either* (i) $\lambda^\omega < \varkappa$ *or* (ii) $\lambda < \varkappa$ *and $\varkappa$ is of cofinality ω. If X is a space of cardinality $\varkappa$ such that each point has a neighborhood of cardinality $\leqslant \varkappa$, then X is not countably compact.*

THEOREM 3. *If X is a locally compact, countably compact, first countable space, and $|X| > \varkappa$, then X omits $\varkappa$ if either* (i) $\mathfrak{c} < \varkappa < \varkappa^\omega$ *or* (ii) *$\varkappa$ is a singular cardinal of cofinality ω.*

Proof. By a classical result of König, $\mathfrak{c}$ does not have cofinality ω, so that in either case we have $\varkappa < \mathfrak{c}$ or $\varkappa > \mathfrak{c}$.

Now, if Y is a closed subspace of X of cardinality $< \mathfrak{c}$, then Y is countably compact and locally countable: every compact, first countable space of cardinality $< \mathfrak{c}$ is countable ([8], 3.17). Hence, by Lemma 3, $|Y| \neq \varkappa$. If $|Y| > \mathfrak{c}$ and Y is closed, we use the fact that every point of X has a neighborhood of cardinality $\leqslant \mathfrak{c}$: by Arkhangel'skiĭ's theorem ([8], p. 31), every compact first countable space has cardinality $\leqslant \mathfrak{c}$. Then, if $\varkappa > \mathfrak{c}$, Lemma 3 again implies $|Y| \neq \varkappa$.

The converse of Theorem 3 (i) is true; in fact:

LEMMA 4. *If X is a first countable space and $\varkappa = \varkappa^\omega < |X|$, then X does not omit $\varkappa$.*

Proof. The only properties of X that we use are tameness and countable tightness; that is,

$$\bar{A} = \bigcup \{\bar{B}\colon B \subset A, B \text{ is countable}\} \quad \text{for all } A \subset X.$$

Then, if $Y \subset X$ and $|Y| = \varkappa$, then $|\bar{Y}| \leqslant \mathfrak{c} \cdot \varkappa^\omega = \varkappa$.

COROLLARY 2. *If X is a locally compact, countably compact, first countable space, and $\mathfrak{c} \leqslant \varkappa < |X|$, then X omits $\varkappa$ if and only if $\varkappa < \varkappa^\omega$.*

For our next corollary, recall one version of the Singular Cardinals Hypothesis (SCH): if $2^{\mathrm{cf}\varkappa} < \varkappa$, then $\varkappa^{\mathrm{cf}\varkappa} = \varkappa^+$. From this there follows: if $\mathfrak{c} \leqslant \varkappa$, and $\varkappa$ is not a singular cardinal of cofinality ω, then $\varkappa^\omega = \varkappa$. This is shown by induction, using the fact that every countable subset of $\varkappa$ is a subset of some $\alpha < \varkappa$.

COROLLARY 3. *If* CH + SCH (*in particular, if* GCH) *holds and X is a locally compact, countably compact, first countable space, and $\varkappa < |X|$, then X omits $\varkappa$ if and only if $\varkappa$ is a singular cardinal of cofinality ω.*

Theorem 3 and these last two corollaries may be vacuous in some models: it is even an unsolved problem whether there is a locally compact, first countable, countably compact space of cardinality $> \mathfrak{c}$ or a locally countable, locally compact, countably compact space of cardinality $> \omega_\omega$ in every model of set theory. However, we now know that any "exceptional" model must have an inner model with measurable cardinals.

EXAMPLE 2. If GCH + $\square_\varkappa$ is true for every singular $\varkappa$ of cofinality ω, there is a locally compact, locally countable, countably compact space of any cardinal λ except for singular cardinals of cofinality ω. In [10] it is shown how to construct such examples, using $\square_\varkappa$ to push the induction up above $\varkappa$ when $\varkappa$ is singular of cofinality ω. The second-named author has recently observed [13] that this construction is still valid if one replaces GCH by the axiom that the cofinality of the poset $\langle [\varkappa]^\omega, \subset \rangle$ is $\varkappa^+$ for all singular cardinals $\varkappa$ of cofinality ω; that is, there is a set $\boldsymbol{B}$ of $\varkappa^+$ countable subsets of $\varkappa$ such that, for every countable $A \subset \varkappa$, there exists $B \in \boldsymbol{B}$ such that $A \subset B$. Unlike "$\varkappa^\omega = \varkappa^+$", this axiom is not destroyed by ccc forcing; more strongly, it holds in V whenever it holds in an inner model M such that Covering (V, M) holds [13], and so does $\square_\varkappa$ for singular $\varkappa$ ([3], Section 8). The core model K is such an M, confirming the comment made prior to this example.

THEOREM 4. *If* "$\mathrm{cf}([\varkappa]^\omega) = \varkappa^+$" *and* $\square_\varkappa$ *are true for all singular cardinals $\varkappa$ of cofinality ω, then for each such cardinal $\varkappa$ there is a hereditarily atomic Boolean algebra of cardinality greater than $\varkappa$ but no homomorphic image of cardinality $\varkappa$.*

Proof. Take the Stone algebra (algebra of clopen subsets) of $X+1$, where $|X| > \varkappa$ and X is locally compact, locally countable, and countably compact. Then $|\mathscr{S}(X+1)| > \varkappa$, and every homomorphic image of $\mathscr{S}(X+1)$ is the Stone algebra of some closed subspace of X of the same cardinality.

The cardinal $\mathrm{cf}([\varkappa]^\omega)$, which will be here denoted by $\varkappa^\#$ for convenience, is an interesting concept, with the help of which we can extend Theorem 3 (ii) and obtain some results on omission of cardinals $< \mathfrak{c}$. We begin with a set-theoretic lemma, pointed out to the second-named author by J. Baumgartner.

LEMMA 5. *Let $\varkappa$ be a cardinal and let A be a collection of countable subsets of $\varkappa$ such that each infinite subset of $\varkappa$ meets some member of A in an infinite set. Then $|A| \geqslant \varkappa^\#$.*

Proof. Let $\psi\colon \varkappa \to \varkappa^{<\omega}$ be a bijection, where $\varkappa^{<\omega}$ stands for the full $\varkappa$-ary tree of height ω. For each $A \in \mathcal{A}$ let $(\psi A)\downarrow$ stand for $\{\sigma$: there exists $a \in A$ such that $\sigma \subset \psi(a)\}$. Now, every countable subset of $\varkappa$ is a subset of

$$\bigcup \{\operatorname{ran} \sigma\colon \sigma \in (\psi A)\downarrow\} \quad \text{for some } A \in \mathcal{A}.$$

Lemma 5 gives an alternative characterization of $\varkappa^{*}$, as being the least cardinality of a set $\mathcal{A}$ as described. Other elementary facts about $\varkappa^{*}$ are that $\varkappa \leqslant \varkappa^{*} \leqslant \varkappa^{\omega}$ for all uncountable $\varkappa$ and that $\varkappa^{\omega} = \mathfrak{c}\cdot\varkappa^{*}$ for all infinite $\varkappa$. Hence, if $\varkappa \geqslant \mathfrak{c}$, then $\varkappa^{\omega} = \varkappa^{*}$.

LEMMA 6. *If X is a locally compact, first countable space of cardinality $\varkappa$ and $\varkappa < \varkappa^{*}$, then X is not countably compact.*

Proof. By Lemma 3 (i), Arkhangel'skiĭ's theorem, and the above elementary facts, it is enough to consider the case $\varkappa < \mathfrak{c}$. In this case, X is locally countable, and we can assign each point p a countable neighborhood V_p. Let $\mathcal{A} = \{V_p\colon p \in X\}$. By Lemma 4, some infinite subset of X must fail to meet any member of $\mathcal{A}$ in an infinite set, making countable compactness impossible.

COROLLARY 4. *If X is a locally compact, countably compact, first countable space and $|X| > \varkappa$, then X omits $\varkappa$ if $\varkappa < \varkappa^{*}$.*

This corollary actually supersedes Theorem 3: if $\varkappa \geqslant \mathfrak{c}$, then $\varkappa < \varkappa^{*}$ is equivalent to $\mathfrak{c} < \varkappa < \varkappa^{\omega}$, while if $\varkappa$ is a singular cardinal of cofinality ω, then an elementary diagonal argument shows that $\varkappa < \varkappa^{*}$.

The converse of Corollary 4 holds in the case $\varkappa \geqslant \mathfrak{c}$ (see Corollary 2). What about if $\varkappa < \mathfrak{c}$? We can do it for the following special class: a space is called *ω-bounded* if every countable subset has compact closure. The spaces of Example 3 are all ω-bounded, and it is even an unsolved problem whether there is a first countable, countably compact space in every model that is not ω-bounded (for a discussion, see [13]).

Of course, for locally countable spaces, ω-boundedness implies that every countable subset has countable closure. This is a very strong version of tameness; using it as we did tameness in Lemma 4, and substituting $\varkappa^{*}$ for $\varkappa^{\omega}$ everywhere, we get

LEMMA 7. *If X is a locally countable space in which every countable subset has countable closure, and $\varkappa = \varkappa^{*} < |X|$, then X does not omit $\varkappa$.*

THEOREM 5. *If X is a locally countable, ω-bounded space and $|X| > \varkappa$, then X omits $\varkappa$ if and only if $\varkappa < \varkappa^{*}$.*

Proof. If X is as described, it satisfies the hypotheses of Lemma 7 and also Corollary 4, since ω-boundedness implies countable compactness for all spaces and local compactness (and thus first countability) for locally countable spaces.

Now, what about the case where X is countably compact but not ω-bounded? Here, complications can arise even in the case $\varkappa = \omega_1$ and, of course, $\omega_1^{*} = \omega_1$. This is the theme of our final section.

5. More on omitting ω_1. Recently, there have been some interesting results on when locally compact, locally countable, countably compact spaces can omit the regular cardinal ω_1. Of course, CH implies it cannot be omitted (Theorem 1), but these results have to do with models where CH fails. One result is

THEOREM 6 ([4]). *If the Proper Forcing Axiom* (PFA) *holds, then a normal, locally hereditarily Lindelöf, countably compact space is either compact or contains a copy of ω_1.*

As a result, PFA implies that normal, locally countable, countably compact spaces cannot omit ω_1. It is still not known whether "normal" can be dropped or replaced by "regular" (equivalently, "locally compact") here or in Theorem 6 (see [13] or [4]).(¹)

The Proper Forcing Axiom is a strengthening of MA(ω_1) and implies MA when $\mathfrak{c} = \omega_2$. In the light of Theorems 1 and 6 it is natural to ask whether MA implies that a normal (or regular), locally countable, countably compact space cannot omit ω_1. The answer is negative:

EXAMPLE 3. In [14], one of us has constructed models of MA, with $\mathfrak{c}$ anything compatible with MA, in which there are separable, locally compact, locally countable, countably compact spaces of cardinality $\mathfrak{c}$ in which every pair of uncountable (equivalently, noncompact) closed subspaces must meet. It is easy to see that any such spaces must be normal and that every uncountable closed subspace must be of cardinality $\mathfrak{c}$ (just cover it with countable open sets and look at the complement of their union). Hence, if $\mathfrak{c} > \omega_1$, such spaces omit ω_1 (and any uncountable cardinal $< \mathfrak{c}$).

Actually, all the MA examples in [14] have this last property. The reason is that they are designed so that every ω-bounded subspace is compact. Now, recall [2] that MA implies $\mathfrak{p} = \mathfrak{c}$ and consider

THEOREM 7. *Let X be a locally compact, locally countable, countably compact space. If X omits ω_1, then every ω-bounded subspace of X is compact. If every ω-bounded subspace of X is compact, then X omits every uncountable cardinal $< \min\{\mathfrak{p}, |X|\}$.*

Proof. Suppose X has an ω-bounded noncompact (hence uncountable) subspace Y. By first countability of X, Y is closed in X. By Lemma 7, Y does not omit ω_1, and neither does X.

Now, suppose X does not omit some uncountable cardinal $\varkappa$. Let Y be a closed subspace of X, of cardinality $\varkappa$. Every separable, regular, countably compact, noncompact space has cardinality $\geqslant \mathfrak{p}$ (see [6]). Hence, if $\mathfrak{p} > \varkappa$, every countable subset of Y has compact closure, i.e., Y is ω-bounded. But Y is not compact, because it is uncountable and locally countable.

(¹) Added in proof. The answer is "yes": PFA implies that every first countable, countably compact space is either compact or contains a copy of ω_1.

We close with some results which illustrate how the cardinality of X can make a difference. The first one shows that Examples 1 and 3 have to be of cardinality $\geqslant \mathfrak{c}$ under MA.

THEOREM 8. *Assume* MA. *If X is locally compact and $|X| < \mathfrak{c}$, then X does not omit ω_1.*

Proof. Because of the conditions on X, it is scattered. Suppose X is locally countable. If X has an ω-bounded noncompact subspace, then (as above) it has one of cardinality ω_1, which is closed in X. Otherwise (see [14]), the one-point compactification of X has countable tightness and, by MA, X is the union of countably many closed discrete subspaces [1], so X does not omit *any* cardinals.

If X is not locally countable, let α be the least ordinal for which there exists $p \in I_\alpha(X)$ such that p has no countable neighborhood, and let $\bar{U}$ be a compact neighborhood of p such that

$$\bar{U} \setminus \{p\} \subset \{I_\beta(X) \colon \beta < \alpha\}.$$

Then $\bar{U} \setminus \{p\}$ is a locally compact, locally countable scattered space, and if K is closed in $\bar{U} \setminus \{p\}$, then $K \cup \{p\}$ is closed in $\bar{U}$, hence in X. By the previous paragraph, $\bar{U} \setminus \{p\}$ does not omit ω_1, and $\bar{U}$ is uncountable, so X does not omit ω_1 •

COROLLARY 5. [MA] *If B is an uncountable hereditarily atomic Boolean algebra of cardinality $< \mathfrak{c}$, then B has a homomorphic image of cardinality ω_1.*

PROBLEM 3 (**P 1365**). Can the assumption of MA be dropped in Theorem 8? Corollary 5?

PROBLEM 4 (**P 1366**). Can we replace ω_1 with any other cardinal $< |X|$ in Theorem 8? Corollary 5?

In contrast to Theorem 8 and Corollary 5, MA by itself places no *upper* restriction on the size of X for omitting ω_1. In [14] there is a construction of "arbitrarily large" locally compact, locally countable, countably compact spaces, compatible with MA, such that every ω-bounded subspace is compact; by Theorem 7 and $\mathfrak{p} = \mathfrak{c}$, such a space omits all uncountable cardinals $< \mathfrak{c}$.

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